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CERTAIN PROBLEMS IN THE THEORY OF STABILITY OF MOTION IN THE WHOLE

by V. A. Pliss

Izdatel'stvo Leningradskogo Universiteta, 1958

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CERTAIN PROBLEMS IN THE THEORY OF STABILITY OF MOTION IN THE WHOLE

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ABSTRACT

In this monograph, certain nonlinear systems of three differential equations are considered. It is assumed that the nonlinearities entering into the system satisfy the generalized Hurwitz conditions. Sufficient conditions for stability in the whole and conditions for which the systems considered have periodic solutions are also considered. Conditions are imposed on the parameters of the systems which are necessary and sufficient for stability in the whole for any nonlinearity.

This work can be useful to specialists in the qualitative theory of differential equations and the theory of automatic control.

FOREWORD

Beginning in 1949, many mathematicians became interested in the problem of Ayzerman. This problem consists of the following.

A system of differential equations is given

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + b(x_k)x_k, \quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j \quad (i = 2, 3, \ldots, n), \tag{*}$$

where $b(x_k) = \frac{f(x_k)}{x_k}$, and $f(x_k)$ represents a function which is continuous in the interval $-\infty < x_k < +\infty$, with $f(0) = 0$, such that for system (*) the fulfillment of the Hurwitz conditions is the negativeness of the roots of the characteristic equation (in which the coefficient of x_k in the first equation is $a_{1k} + b(x_k)$). It is asked whether the trivial solution $x_1 = x_2 = \ldots = x_n = 0$ is asymptotically stable[1] in the whole (as, obviously, it would be for constant $b(x_k)$).

This question is completely solved for the case n = 2. In this case it was shown that for certain a_{kl}, asymptotic stability in the whole of the solution $x_1 = x_2 = 0$ takes place independently of the supplementary properties of $f(x_k)$. However, in the presence of one such relationship among the a_{kl}, there is asymptotic stability in the whole of the null solution only for certain supplementary conditions on the $f(x_k)$, which are accurately determined. However, if these supplementary conditions on $f(x_k)$ are not fulfilled, then there is a domain of stability bounded by motions which go to infinity as $t \to +\infty$ and as $t \to -\infty$. These boundaries of the motion can be found with any degree of precision. Qualitative illustrations of the behavior of such systems are also studied and appear sufficiently simple.

Thus, in the case of two equations, Ayzerman's system was well studied, and for such systems, stability in the whole for certain relations between the a_{kl} is destroyed by the presence of motions which go to infinity at both end points in time. It is noted that in the general case of a system of two equations, a Lyapunov function can be found which, while it does not settle the stability question of the null solution, still eliminates the existence of a periodic solution.

For a system of three differential equations, there are no Lyapunov functions in the general case. For systems of three equations of the Ayzerman type, only sufficient conditions for the global stability of the null solution were found, which were established on the basis of Lyapunov function construction.

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[1]See the definition of asymptotic stability in the whole on page 7.

First considered in this monograph are such systems of three equations of the Ayzerman type where the matter of global stability of the null solution is established by more exact methods of analysis. Systems are also shown in which the global stability of the null solution is violated by the existence of periodic solutions. These periodic solutions are discovered here by the properties of field direction which are qualitatively different from those which, until now, employed the establishment of the existence of periodic solutions of three-equation systems. The reader is referred to the monograph itself for other interesting, distinguished points of its contents.

The problems investigated in this work are difficult, and efforts must be turned to their solution. For the solution of these questions, the author applies new investigational methods which enable the discovery of the existence of periodic solutions and the matter of global stability of the null solution. Apparently, these new methods are also useful for examining other systems of differential equations.

The author evidently concurs with the following statement made by P. L. Chebyshev concerning the words of A. M. Lyapunov:

> "To be engaged in easy, although also new, problems which can be resolved by well-known methods is worth nothing, and therefore every young scholar, if he already has some practice in the solution of mathematical problems, must try his skill on some serious problem which exhibits known theoretical difficulty."

The attentive reader will undoubtedly continue his interest in the investigations of V. A. Pliss. Already in this monograph there are, possibly, prepared solutions to problems about important generalizations of the conditions for the existence of periodic solutions. Indistinctly, a conjecture on the monotonicity of the nonlinearity at infinity for the boundedness of all solutions of the systems considered is evoked by the raising of a question or by the method of investigation (chapter V of the monograph).

N. P. Yerugin

INTRODUCTION

In this book, a special case of a system of three differential equations of the Ayzerman type (ref. 1) is studied. The problem as posed by Ayzerman is formulated in the following way.

Given the system of linear differential equations:

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + Fx_k, \quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j \quad (i=2, 3, \ldots, n). \tag{1}$$

For given constants a_{pj}(pj = 1, 2, . . . , n) and for any value F in some interval $\alpha < F < \beta$, let all the roots of the characteristic equation of system (1) have negative real parts. This is required to prove or disprove the following assertion.

For any interval (α, β) for which, with $\alpha < F < \beta$, the roots of the characteristic equation of system (1) still maintain negative real parts, and for any single-valued continuous function f(x) satisfying the conditions:

$$\alpha x^2 < f(x)x < \beta x^2 \text{ for all } x \neq 0 \tag{a}$$

$$f(0) = 0, \tag{b}$$

and the system:

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + f(x_k), \quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j \quad (i=2, 3, \ldots, n), \tag{2}$$

at its unique equilibrium state, which obviously corresponds to the origin of coordinates $x_1 = x_2 = \ldots = x_n = 0$, will have at the origin a stable equilibrium state, and the domain of attraction includes the entire phase space of the system; i.e. $-\infty < x_j < \infty$ (j = 1, 2, . . . , n).

In the following, we will call the function f(x) the nonlinearity, and conditions (a) and (b) the generalized Hurwitz conditions.

Ayzerman's problem has been resolved in a negative sense; that is, N. N. Krasovskiy (ref. 2) constructed an example in which the null solution of system (2) is not globally stable. However, there is still interest in the following questions. For what values of the parameters a_{pj} entering into system (2) will the null solution of this system be globally stable for any nonlinearity satisfying the generalized Hurwitz conditions? If the null solution of system (2) is not globally stable for any nonlinearity satisfying the generalized Hurwitz conditions, then what additional restriction on the nonlinearity is required for the null solution to be globally stable? For what values of the parameters a_{pj} and for what nonlinearities will the null solution not be globally stable?

Answers to all these questions are available for systems of two differential equations of the Ayzerman type. We consider the system:

$$\frac{dx}{dt} = ay + f(x), \quad \frac{dy}{dt} = bx + cy, \tag{3}$$

where a, b and c are constants, and the continuous function $f(x)$ is single value and satisfies the generalized Hurwitz conditions:

$$\left.\begin{array}{c} x\,[f(x) + cx] < 0, \quad x\,[cf(x) - abx] > 0 \\ \text{for } x \neq 0 \text{ and } f(0) = 0 \end{array}\right\} \tag{4}$$

If $c^2 + ab \neq 0$, then, as shown by N. P. Yerugin (refs. 3, 4, and 5), the null solution of system (3) is globally stable for any nonlinearity $f(x)$ satisfying the generalized Hurwitz conditions (GHC) (4). N. P. Yerugin (ref. 5) proved that, for the null solution of system (3) to be globally stable in the case $c^2 + ab = 0$, it is sufficient that (5) be fulfilled:

$$\left.\begin{array}{l} \overline{\lim\limits_{x \to +\infty}} \left[\int\limits_0^x (cf(x) - abx)\,dx + cf(x) - abx \right] = +\infty \\ \overline{\lim\limits_{x \to -\infty}} \left[\int\limits_0^x (cf(x) - abx)\,dx - cf(x) + abx \right] = +\infty \end{array}\right\} \tag{5}$$

N. N. Krasovskiy (ref. 6) proved that condition (5) is also necessary for the global stability of the null solution of system (3) whenever $c^2 + ab = 0$. Reference 7 explains that for $c^2 + ab = 0$ a curve can be constructed bounding the domain of attraction for the point $x = y = 0$, when the null solution is not globally stable. But this monograph studies in detail the qualitative picture of the integral curves behavior inside the domain as well as outside its boundaries.

I. G. Malkin's work (ref. 8) shows that for system (3) under conditions (4) there always exists a Lyapunov function in the form of "an integral of the nonlinearity plus a quadratic form of the coordinates of the phase space" (which, otherwise, does not always guarantee the global stability of the null solution). Therefore, two-equation systems of the Ayzerman type do not admit periodic motions different from the position of equilibrium. Moreover, from (5), it follows that in the plane case of Ayzerman's problem, the resolution of the global stability problem does not depend on the nonlinearity behavior in a limited portion of the real axis. Many authors (refs. 9–17, and 35–40) studied different, special cases of three-differential-equation systems of the Ayzerman type (a detailed summary of such works with the resulting equations is included in a survey paper by N. P. Yerugin (ref. 18)). In all of these works, the authors attempted to construct Lyapunov functions in the form of an "integral of the nonlinearity plus a quadratic form," and with its help they obtained some sufficient conditions for global stability.

In addition to the cited references, the present work studies Ayzerman-type systems of three differential equations. In system (2) it is assumed that $k = 1$, $n = 3$, and the nonlinearity $f(x)$ satisfies the GHC. The basic problem solved in this work consists of the following: for what values of the parameters a_{ij} is the null solution of (2) globally stable for any function $f(x)$ satisfying the GHC? In other words, the problem is resolved for which a_{ij} the place has a positive answer on the question of Ayzerman's problem. For the case when $n = 3$, $k = 1$, this problem is completely solved; i.e., conditions are given which are necessary and sufficient for global stability with any nonlinearity. At the same time, a series of other questions are resolved in the work. For the special case $a_{22} + a_{33} = 0$, a more detailed analysis is undertaken than in the general case $a_{22} + a_{33} \neq 0$.

Chapter I establishes a general theorem on the global stability of motion which is used in later discussions.

The studied system is brought into a certain special form in chapter II. In the same place, the GHC is written in a clear form, and various special cases are considered.

Chapter III establishes several theorems on the general characteristics of the arrangement of the trajectories of the system studied. In particular, it explains how, for certain supplementary conditions imposed on the coefficients a_{ij}, the solutions of system (2) are related to individual oscillating figures.

A series of sufficient conditions are given in chapter IV for global stability for any nonlinearity satisfying the GHC. In those cases when global stability for any nonlinearity cannot be established, conditions are formulated to be imposed on the nonlinearity which are sufficient for global stability of the null solution.

For the proofs of many theorems in chapter IV, Lyapunov functions are used in the form of an "integral of the nonlinearity plus a quadratic form of the sought-after functions." A method is given which permits establishing conditions necessary and sufficient for the existence in system (2) of a Lyapunov function of such special form. The theorem demonstrated in chapter I is used for the proofs of other theorems in chapter IV.

The special case $a_{22} + a_{33} = 0$ is specifically studied in chapters V and VI. Conditions sufficient for the boundedness of all the solutions of the system studied are formulated. For the fulfillment of these conditions, the qualitative behavior of the trajectories is made more precise. In particular, the following alternative assertion is proven: for the conditions defined, either the null solution is globally stable or system (2) has a periodic solution.

Chapter VI states the conditions sufficient for the presence in system (2) of periodic solutions different from the equilibrium position. To prove that such solutions exist in the system studied, a method is used which is different from the principle of the transformation of a linear element into itself used by A. A. Andronov and A. G. Mayer (ref. 19); the principle of the torus used by K. O. Fridrichs (ref. 20), L. M. Bauch (ref. 21) and V. V. Nemytskiy (ref. 22); and the principle of transforming the entire space into itself used by

G. Colombo (ref. 34). (See also the survey paper of V. V. Nemytskiy (ref. 23) on the periodic solutions of systems of three differential equations.)

Sufficient conditions are given in chapter VII for the absence of global stability in the case $a_{22} + a_{33} \neq 0$. In this general case, instability can be realized by the appearance of periodic solutions as well as by the appearance of solutions going to infinity for the continual variation of all the sought-after functions (just as in the degenerate case of the Ayzerman plane problem).

In the conclusions, conditions are formulated which are necessary and sufficient for the global stability of the null solution of the system studied for any nonlinearity satisfying the GHC.

For the completion of this work, invaluable help and assistance was extended by my teacher N. P. Yerugin, whom I warmly and sincerely thank. I give my deepest thanks to V. I. Smirnov for the extremely valuable references to my work. To V. P. Basov I express my deepest appreciation for reading the manuscript and making a series of important critical remarks which were accepted for the ultimate preparation of the work for the printers.

Chapter I. A GENERAL THEOREM

Section 1

We consider the system of ordinary differential equations,

$$\frac{dx_s}{dt} = X_s(x_1, x_2, \ldots, x_n) \qquad (s = 1, 2, \ldots, n), \tag{1.1}$$

the right sides of which are defined and continuous and satisfy the conditions for uniqueness of solutions for any real $x_1, x_2, \ldots, x_n$. In addition, we assume that

$$X_s(0, 0, \ldots, 0) = 0 \qquad (s = 1, 2, \ldots, n). \tag{1.2}$$

We will say that the null solution of system (1.1) is globally stable if it is Lyapunov stable and if, along all solutions of system (1.1), the following condition holds:

$$\lim_{t \to +\infty} x_s(t) = 0 \qquad (s = 1, 2, \ldots, n). \tag{1.3}$$

By $\phi(p, t)$ we will designate that trajectory of system (1.1) which, for $t = 0$, passes through the point p.[1]

Consider the hyperplane

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \tag{1.4}$$

of the phase space. We say that trajectory $\phi(p, t)$ of system (1.1) intersects the hyperplane (1.4) for $t = t_1$ if instants of time, $t_0 < t_1 \leq t_2 < t_3$, exist such that on trajectory $\phi(p, t)$ the following conditions are fulfilled:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \qquad \text{for} \quad t \in [t_1, t_2] \tag{1.5}$$

and

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \neq 0 \tag{1.6}$$

[1]Throughout this book, ϕ appears in equations within the text while φ may appear in equations set apart from lines of text. Both symbols represent the same quantity.

for

$$t \epsilon [t_0, t_1) \text{ and } t \epsilon (t_2, t_3].$$

For this, the signs of the expression $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ on the trajectory $\phi(p, t)$ in the intervals $[t_0, t_1)$ and $(t_2, t_3]$ are different. In other words, at the intersection the trajectory goes from one side of the hyperplane of (1.4) to the other. For this, the trajectory $\phi(p, t)$ can lie in the hyperplane (1.4) for a finite interval of time (if $t_1 < t_2$), or have with it only one common point (if $t_1 = t_2$). In both cases the point of intersection of the trajectory $\phi(p, t)$ with the hyperplane (1.4) will be called the point $\phi(p, t_1)$.

We have the following theorem.

Theorem 1.1

For the trivial solution of system (1.1) to be globally stable, it is necessary and sufficient for the following conditions to be fulfilled:

(1) The point $(0, 0, \ldots, 0)$ is an isolated equilibrium point of the system (1.1).

(2) The equilibrium position $(0, 0, \ldots, 0)$ is *stable in the sense of* Lyapunov.

(3) There exists a hyperplane L of the type (1.4) such that:

(a) along any trajectory for which a T can be chosen such that for $t > T$, the trajectory does not intersect the hyperplane L, then $x_S \to 0$ $(s = 1, 2, \ldots, n)$ for $t \to +\infty$;

(b) there exists a continuous function v defined only for points on the hyperplane L which has the qualities $v(0, 0, \ldots, 0) = 0$, $v(x_1, x_2, \ldots, x_n) > 0$ for $(x_1, x_2, \ldots, x_n) \epsilon L$ and $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$, $v(x_1, x_2, \ldots, x_n) \to \infty$ for $(x_1, x_2, \ldots, x_n) \epsilon L$ and $\sum_{i=1}^{n} x_i^2 \to \infty$;

(c) if a trajectory $\phi(p, t)$ of system (1.1) has at least two points in common with the hyperspace L, then the following is fulfilled: let t_1 and t_2 be two instants of time for which the points $\phi(p, t_1)$ and $\phi(p, t_2)$ lie in the hyperplane L; then, if $t_1 < t_2$, $v(\phi(p, t_1)) > v(\phi(p, t_2))$.

This theorem with a short proof is in reference 24; here, a detailed proof is shown.

Proof

1. Sufficiency

Because of condition 3a, it is sufficient to prove only the following fact. Any trajectory $\phi(p, t)$ which possesses the quality that increasing sequence of instants of time can be chosen,

$$t_1, t_2, t_3, \ldots \to +\infty, \tag{1.7}$$

such that the points $\phi(p, t_k)$ are points of intersection of the trajectories $\phi(p, t)$ with the hyperplane L, goes toward the origin of coordinates for $t \to \infty$. We will now prove this.

By virtue of condition 3c, all $\phi(p, t_k)$ lie in the domain:

$$v \leq v(\phi(p, t_1)). \tag{1.8}$$

Domain (1.8), because of condition 3b, is bounded. Therefore, according to the Bolzano-Weirstrasse theorem, a sequence of points $\phi(p, t_k)$ can be found which converge at point q; i.e.,

$$\lim_{k \to \infty} \varphi(p, t_k) = q. \tag{1.9}$$

In consequence of (1.7), point q is an ω-limit point for trajectory $\phi(p, t)$. We prove that if at the instant of time τ, trajectory $\phi(p, t)$ intersects hyperplane L, then:

$$v(\phi(p, \tau)) > v(q). \tag{1.10}$$

As a result of condition 3c of the theorem, the sequence $v(\phi(p, t_k))$ is decreasing. Moreover, because of the continuity of function v and relation (1.9), we have

$$\lim_{k \to \infty} v(\varphi(p, t_k)) = v(q). \tag{1.11}$$

Consequently,

$$v(\phi(p, t_k)) > v(q) \tag{1.12}$$

for all k. Resulting from (1.7), an i can be found such that $t_i > \tau$; and, necessarily, due to condition 3c,

$$v(\phi(p, \tau)) > v(\phi(p, t)). \tag{1.13}$$

Therefore, (1.10) follows from (1.12).

We will now prove that q coincides with the origin. Indeed, assume to the contrary that $q \neq (0, 0, \ldots, 0)$, and consider a trajectory $\phi(q, t)$ of system (1.1). For $\phi(p, t)$, this trajectory is an ω-limit point. Also assume at first that a $t^* > 0$ exists such that for $t = t^*$, $\phi(q, t)$ intersects hyperplane L. From condition 3c of the theorem we have

$$v(\phi(q, t^*)) < v(q). \tag{1.14}$$

Therefore, by using the theorem on integral continuity and the definition of the intersection of a trajectory of system (1.1) with the hyperplane of type (1.4) which was stated above, we can choose τ such that trajectory $\phi(p, t)$ intersects hyperplane L for $t = \tau$ and $v(\phi(p, \tau)) < v(q)$. But this contradicts inequality (1.10). Consider now the case when trajectory $\phi(p, t)$ does not intersect hyperplane L for $t > 0$. Then, from condition

3a, $\phi(q, t) \to (0, 0, \ldots, 0)$ for $t \to \infty$, and, consequently, the origin of the coordinates is the ω-limit point for trajectory $\phi(p, t)$. But from here, because of condition 2 of the theorem, $\phi(p, t)$ goes to the position of equilibrium $(0, 0, \ldots, 0)$, which contradicts $q \neq (0, 0, \ldots, 0)$ being the ω-limit point for $\phi(p, t)$. The contradictions obtained show that point q coincides with the origin, and, consequently, trajectory $\phi(p, t)$ has as its ω-limit point the Lyapunov stable position of equilibrium $(0, 0, \ldots, 0)$. Therefore, trajectory $\phi(p, t)$ goes to the origin as $t \to \infty$.

The sufficiency of the conditions has been proven.

2. Necessity

Let the trivial solution of system (1.1) be globally stable, and take an arbitrary hyperplane of type (1.4). Then conditions 1, 2 and 3a are obviously fulfilled. Now only the existence of a function v which satisfies the conditions of 3c and 3b must be proven. As proved by Ye. A. Barbashin and N. N. Krasovskiy (ref. 25), from the conditions of the theorem, there exists a continuously differentiable, positive definite, infinitely large function w, the derivative of which, by virtue of the equations of system (1.1), is negative definite. It is not difficult to see, then, that the function w satisfies all the conditions of the theorem, if it is considered as a function of the points on hyperplane L. Thus, the theorem is proven.

Chapter II. SOME CERTAIN TRANSFORMATIONS

Section 2

In the present work, a system of three differential equations of the Ayzerman type is studied (ref. 1). The system considered has the following form:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(x) + a_{12}y_1 + a_{13}z_1 \\ \frac{dy_1}{dt} &= a_{21}x + a_{22}y_1 + a_{23}z_1 \\ \frac{dz_1}{dt} &= a_{31}x + a_{32}y_1 + a_{33}z_1 \end{aligned}\right\} \tag{2.1}$$

We assume that

$$\Delta_{11} = \begin{vmatrix} a_{22}, & a_{23} \\ a_{32}, & a_{33} \end{vmatrix}, \quad \Delta_{21} = -\begin{vmatrix} a_{12}, & a_{13} \\ a_{32}, & a_{33} \end{vmatrix}, \quad \Delta_{31} = \begin{vmatrix} a_{12}, & a_{13} \\ a_{22}, & a_{33} \end{vmatrix}. \tag{2.2}$$

In the following, we will assume that

$$\Delta_{21}^2 + \Delta_{31}^2 \neq 0. \tag{2.3}$$

The case $\Delta_{21}^2 + \Delta_{31}^2 = 0$ is of no interest to us for the following reason. If $a_{12} = a_{13} = 0$, then system (2.1) is integrable by quadratures. If $\Delta_{21} = \Delta_{31} = 0$, but $a_{12}^2 + a_{13}^2 \neq 0$, then system (2.1) can be written in the following way:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(x) + a_{12}y_1 + a_{13}z_1 \\ \frac{dy_1}{dt} &= a_{21}x + (a_{12}y_1 + a_{13}z_1)\,n_1 \\ \frac{dz_1}{dt} &= a_{31}x + (a_{12}y_1 + a_{13}z_1)\,n_2 \end{aligned}\right\}, \tag{2.4}$$

where n_1 and n_2 are constants combined in an appropriate way. However, it is seen immediately that every one of the points lying on the line $x = 0$, $a_{12}y_1 + a_{13}z_1 = 0$ is an equilibrium position for system (2.4) and, consequently, also for system (2.1). But in this work it is assumed that function $f_1(x)$ satisfies the GHC, and, as a result system (2.1) cannot have equilibrium positions different from the origin. Without losing generality, we can assume that

$$\Delta_{21} \neq 0. \tag{2.5}$$

The generality is not lessened by this assumption since the case $\Delta_{31} \neq 0$ is brought into the case $\Delta_{21} \neq 0$ by a simple relabeling of the variables. Further, we will assume that

$$\frac{a_{13}}{a_{12}} \neq \frac{\Delta_{31}}{\Delta_{21}}. \tag{2.6}$$

In the case when inequality (2.6) is not fulfilled, system (2.1) was investigated by N. N. Krasovskiy (ref. 6), who demonstrated necessary and sufficient conditions in this case for the global stability of the null solution.

For function $f_1(x)$, we will assume that it is defined and continuous, and satisfies the conditions for a unique solution for all real x. In addition, we will assume that $f_1(x)$ satisfies the GHC.

On system (2.1) the following transformation of variables is performed:

$$y = -(a_{22} + a_{33})x + a_{12}y_1 + a_{13}z_1,$$
$$z = \Delta_{11}x + \Delta_{21}y_1 + \Delta_{31}z_1.$$

Because of inequality (2.6), the transformation is nonsingular. Note that this transformation shows only some of the variables of the transformation proposed by N. N. Krasovskiy in reference 6. In the new variables, system (2.1) has the following form:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(x) + (a_{22} + a_{33})x + y \\ \frac{dy}{dt} &= -(a_{22} + a_{33})f_1(x) + (a_{12}a_{21} + a_{13}a_{31} - \Delta_{11})x + z \\ \frac{dz}{dt} &= \Delta_{11}f_1(x) + (a_{21}\Delta_{21} + a_{31}\Delta_{31})x \end{aligned}\right\} \tag{2.7}$$

We assume that

$$f(x) = -f_1(x) - (a_{22} + a_{33})x,$$

$$a = -(a_{21}\Delta_{21} + a_{31}\Delta_{31}) + (a_{22} + a_{33})\Delta_{11},$$

$$b = \Delta_{11},$$

$$c = -(a_{12}a_{21} + a_{13}a_{31} - \Delta_{11}) - (a_{22} + a_{33})^2,$$

$$d = -(a_{22} + a_{33}).$$

Then system (2.7) takes the following form:

$$\left.\begin{aligned} \frac{dx}{dt} &= y - f(x) \\ \frac{dy}{dt} &= z - cx - df(x) \\ \frac{dz}{dt} &= -ax - bf(x) \end{aligned}\right\} \tag{2.8}$$

We assumed that function $f_1(x)$ satisfies the GHC, but at the same time, function f(x) also satisfies this condition. Considering f(x)/x as a constant quantity, we have for the characteristic equation of system (2.8):

$$\begin{vmatrix} -\frac{f(x)}{x} - \lambda, & 1, & 0 \\ -c - d\frac{f(x)}{x}, & -\lambda, & 1 \\ -a - b\frac{f(x)}{x}, & 0, & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^3 + \frac{f(x)}{x}\lambda^2 + \left(c + d\frac{f(x)}{x}\right)\lambda + \left(a + b\frac{f(x)}{x}\right) = 0.$$

Therefore, the GHC for system (2.8) will have the following form:

$$f(0) = 0, \tag{2.9}$$

$$\frac{f(x)}{x} > 0 \qquad \text{for} \quad x \neq 0, \tag{2.10}$$

$$a + b\frac{f(x)}{x} > 0 \qquad \text{for} \quad x \neq 0, \tag{2.11}$$

$$d\frac{f^2(x)}{x} + (c - b)\frac{f(x)}{x} - a > 0 \qquad \text{for} \quad x \neq 0. \tag{2.12}$$

System (2.8) with conditions (2.9)–(2.12) will also be investigated in the following. For this, the special case d = 0 will be subjected to somewhat more detailed analysis than will the general case d ≠ 0. Concerning this, in the case d = 0 we make the following substitutions. From the GHC, (2.10)–(2.12), it follows that c > 0 for d = 0. Having noted this, we set

$$y = \sqrt{c}\,y_2, \quad z = cz_2, \quad t = \frac{t_2}{\sqrt{c}}. \tag{2.13}$$

Then system (2.8) takes on the form

$$\frac{dx}{dt_2} = y_2 - \frac{1}{\sqrt{c}} f(x); \quad \frac{dy_2}{dt_2} = z_2 - x;$$

$$\frac{dz_2}{dt_2} = -\frac{a}{c\sqrt{c}} x - \frac{b}{c\sqrt{c}} f(x). \tag{2.14}$$

Here we set

$$\frac{1}{\sqrt{c}} f(x) = f_2(x); \quad \frac{a}{c\sqrt{c}} = a_2; \quad \frac{b}{c} = b_2$$

Dropping the index 2, we arrive at the following system of differential equations:

$$\frac{dx}{dt} = y - f(x); \quad \frac{dy}{dt} = z - x; \quad \frac{dz}{dt} = -ax - bf(x). \tag{2.15}$$

As follows from relations (2.9)–(2.12), the GHC for system (2.15) has the form below:

$$f(0) = 0, \tag{2.16}$$

$$\frac{f(x)}{x} > 0 \qquad \text{for} \quad x \neq 0, \tag{2.17}$$

$$a + b\frac{f(x)}{x} > 0 \qquad \text{for} \quad x \neq 0 \tag{2.18}$$

$$\frac{f(x)}{x} - \left(a + b\frac{f(x)}{x}\right) > 0 \qquad \text{for} \quad x \neq 0. \tag{2.19}$$

Depending on parameters a, b, c and d of system (2.8), the GHC (2.9)–(2.12) for function f(x) is written in different ways. In connection with this, an entire series of different cases arises. We assume at first that d = 0; i.e., first we consider system (2.15). Then inequalities (2.17), (2.18) and (2.19) yield the following cases:

(1) $0 < b < 1$, $a > 0$

$$\frac{f(x)}{x} > \frac{a}{1-b} \qquad \text{for} \quad x \neq 0. \tag{2.20}$$

(2) $0 < b < 1$, $a = 0$

$$\frac{f(x)}{x} > 0 \qquad \text{for} \quad x \neq 0. \tag{2.21}$$

(3) $0 < b < 1$, $a < 0$

$$\frac{f(x)}{x} > -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.22}$$

(4) $b = 0$, $a > 0$

$$\frac{f(x)}{x} > a \qquad \text{for} \quad x \neq 0. \tag{2.23}$$

(5) $b < 0$, $a > 0$

$$\frac{a}{1-b} < \frac{f(x)}{x} < -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.24}$$

(6) $b = 1$, $a < 0$

$$\frac{f(x)}{x} > -a \qquad \text{for} \quad x \neq 0. \tag{2.25}$$

(7) $b > 1$, $a < 0$

$$-\frac{a}{b} < \frac{f(x)}{x} < \frac{a}{1-b} \qquad \text{for} \quad x \neq 0. \tag{2.26}$$

Now let $d \neq 0$. In this case we set:

$$A = \frac{-(c-b) + \sqrt{(c-b)^2 + 4ad}}{2d},$$

$$B = \frac{-(c-b) - \sqrt{(c-b)^2 + 4ad}}{2d}.$$

These designations have sense when $(c - b)^2 + 4ad \geq 0$. Inequalities (2.10), (2.11) and (2.12) are brought to the following different cases:

(8) $d < 0$, $b > 0$, $A > \max\left\{0, -\frac{a}{b}\right\}$

$$A < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.27}$$

(9) $d < 0$, $b > 0$, $A = -\frac{a}{b} = 0$

$$A < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.28}$$

(10) $d < 0$, $b > 0$, $B > -\frac{a}{b} > \max\{A, 0\}$

$$-\frac{a}{b} < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.29}$$

(11) $d < 0,\ b = 0,\ A > 0$

$$A < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.30}$$

(12) $d < 0,\ b < 0,\ 0 < A < B < -\frac{a}{b}$

$$A < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.31}$$

(13) $d < 0,\ b < 0,\ 0 < A < B = -\frac{a}{b}$

$$A < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.32}$$

(14) $d < 0,\ b < 0,\ 0 < A < -\frac{a}{b} < B$

$$A < \frac{f(x)}{x} < -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.33}$$

(15) $d > 0,\ b > 0,\ (c - b)^2 + 4ad < 0$

$$\frac{f(x)}{x} > -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.34}$$

(16) $d > 0,\ b > 0,\ A > \max\left\{0, -\frac{a}{b}\right\}$

$$\frac{f(x)}{x} > A \qquad \text{for} \quad x \neq 0. \tag{2.35}$$

(17) $d > 0,\ b > 0,\ A = -\frac{a}{b} > 0$

$$\frac{f(x)}{x} > A \qquad \text{for} \quad x \neq 0. \tag{2.36}$$

(18) $d > 0,\ b > 0,\ A = -\frac{a}{b} = 0$

$$\frac{f(x)}{x} > A \qquad \text{for} \quad x \neq 0. \tag{2.37}$$

(19) $d > 0,\ b > 0,\ -\frac{a}{b} > \max\{A, 0\}$

$$\frac{f(x)}{x} > -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.38}$$

(20) $d > 0,\ b > 0,\ B > -\frac{a}{b} > 0$

$$-\frac{a}{b} < \frac{f(x)}{x} < B \qquad \text{for} \quad x \neq 0. \tag{2.39}$$

(21) $d > 0,\ b = 0,\ A > 0$

$$\frac{f(x)}{x} > A \qquad \text{for} \quad x \neq 0. \tag{2.40}$$

(22) $d > 0,\ b < 0,\ 0 < A < -\frac{a}{b}$

$$A < \frac{f(x)}{x} < -\frac{a}{b} \qquad \text{for} \quad x \neq 0. \tag{2.41}$$

Note that this classification of the cases was proposed by A. P. Tuzov in reference 12. But here, only the number of cases and the designation of the coefficients are changed.

A consideration of cases 15 and 19 shows that system (2.8) is brought to a system called indirect regulation by a simple transformation of variables and of function f(x). This system is completely discussed in reference 40 which proves that the null solution of system (2.8) is globally stable for any nonlinear function f(x) satisfying the GHC, (2.34), if the conditions of cases 15 and 19 are fulfilled. Reference 12 shows that the null solution of system (2.8) is globally stable for any f(x) satisying the GHC, (2.21), if the conditions of case 2 are fulfilled. Therefore, cases 2, 15 and 19 are excluded from consideration in the following.

Combine inequality (2.11) with inequality (2.12); then, after dividing by $\frac{f(x)}{x} > 0$, we obtain

$$d\frac{f(x)}{x} + c > 0 \quad \text{for} \quad x \neq 0. \tag{2.42}$$

We will prove that for all cases except 13 and 17, a number $\varkappa > 0$ can be found such that a stronger inequality will take place

$$d\frac{f(x)}{x} + c > \varkappa \quad \text{for} \quad x \neq 0. \tag{2.43}$$

Let the GHC for system (2.8) be written in the form

$$\lambda < \frac{f(x)}{x} < \mu. \tag{2.44}$$

For this, μ can be a nonproper number. Inequality (2.43) could not be fulfilled if, for $\frac{f(x)}{x} = \lambda$ or for $\frac{f(x)}{x} = \mu$, inequalities (2.11) and (2.12) at some time become equalities.

As a direct investigation will show, this is possible only in the conditions of cases 9, 13, 17 and 18. In cases 9 and 18 we will prove that inequality (2.43) is fulfilled for all f(x) satisfying the GHC. As seen from inequality (2.18), it is sufficient for case 9 to prove

$$dB + c > 0. \tag{2.45}$$

But from the conditions of case 9, $dB = -c + b$; therefore, (2.45) follows and, with it, (2.43).

In case 18 it is obviously sufficient to prove that $c > 0$. But from the conditions of case 18, it follows that $-(c - b) + |c - b| = 0$; consequently, $c \geq b > 0$, and thus (2.43) also follows.

Therefore, inequality (2.43) cannot be fulfilled in cases 13 and 17.

Chapter III. THEOREMS ON THE GENERAL CHARACTERISTICS OF THE BEHAVIOR OF THE TRAJECTORIES OF THE SYSTEM STUDIED

This chapter will consider the general characteristics of the behavior of trajectories of system (2.8) when conditions (2.10), (2.11) and (2.12) are fulfilled. By $\phi(p, t)$, we will designate that trajectory of system (2.8) which, for $t = 0$, passes through point p of the phase space. Let E be some set of points of the phase space; then, by $\phi(E, t)$, we will designate the set of those trajectories of system (2.8) which for $t = 0$ passes through points of set E.

Section 3

We describe the field of the directions determined by system (2.8) for conditions (2.10), (2.11) and (2.12). It is seen immediately from system (2.8) that for $y - f(x) > 0$, x increases along all motions of syster (2.8); but for $y - f(x) < 0$, x decreases with increasing time t. For $z > cx + df(x)$, y increases along all the motions of system (2.8); but for $z < cx + df(x)$, y decreases with increasing time. As is seen from inequality (2.11), the derivative $\frac{dz}{dt}$ has a sign opposite the sign of x. Therefore, for $x > 0$, z decreases along all the motions of (2.8), but for $x < 0$, z increases with increasing time.

The trajectories of (2.8) for $y > 0$ intersect the plane $x = 0$, crossing from the half-space $x < 0$ to the half-space $x > 0$; while for $y < 0$, the trajectories of (2.8) intersect the plane $x = 0$, crossing, with increasing time, from the half-space $x > 0$ into the half-space $x < 0$. For $y = 0$, the trajectories of (2.8) are tangent to the plane $x = 0$. Let point p lie on axis Oz, and let its z component be positive. Then trajectory $\phi(p, t)$ of (2.8) is tangent to the plane $x = 0$ at $t = 0$, and for $t \neq 0$, but sufficiently small, the trajectory lies in the half-space $x > 0$. If point p lies on the negative Oz axis, then trajectory $\phi(p, t)$ is tangent to the surface $x = 0$ at point p in such a way that for $t \neq 0$, but sufficiently small, $\phi(p, t)$ lies in the half-space $x < 0$. Thus, the z-axis component on the plane $x = 0$ experiences a maximum for $y > 0$ and a minimum for $y < 0$ along all motions of system (2.8). For $y = 0$, the z-axis component on plane $x = 0$ does not have an extremur along the trajectories of (2.8).

We will now describe the behavior of the trajectories of system (2.15) in the neighborhood of plane $z - x = 0$. For this we will consider $x \geq 0$ since for $x < 0$ the situation will obviously be the same. Let point p lie in plane $z - x = 0$. If at point p, $y \geq f(x)$, then trajectory $\phi(p, t)$ intersects plane $z - x = 0$ for $t = 0$, crossing from the half-space $z - x > 0$ into the half-space $z - x < 0$.

If at point p, $y - f(x) < 0$, but

$$\frac{dz}{dx} = \frac{-ax - bf(x)}{y - f(x)} > 1, \tag{3.1}$$

then trajectory $\phi(p, t)$ at point p intersects plane $z - x = 0$ as well as crossing from the half-space $z - x > 0$ into the half-space $z - x < 0$. In both these cases, the y component of trajectory $\phi(p, t)$ intersecting the y axis assumes a maximum at point p.

Now, at point p, let $x \geq 0$, $z - x = 0$, $y - f(x) < 0$ and

$$\frac{dz}{dx} = \frac{-ax - bf(x)}{y - f(x)} < 1, \tag{3.2}$$

Then trajectory $\phi(p, t)$ intersects plane $z - x = 0$, crossing from half-space $z - x < 0$ into half-space $z - x > 0$. The ordinate of trajectory $\phi(p, t)$ takes on a minimum. But if $y - f(x) < 0$ and

$$\frac{dz}{dx} = \frac{-ax - bf(x)}{y - f(x)} = 1, \tag{3.3}$$

then trajectory $\phi(p, t)$ is tangent to plane $z - x = 0$ at point p.

Now let point p lie on the cylindrical surface $y - f(x) = 0$, for which it is assumed, as before, that $x \geq 0$ at point p. If at point p, $z - cx - df(x) > 0$, then $\phi(p, t)$ intersects the surface $y - f(x) = 0$ at point p, crossing from domain $\{y - f(x) < 0\}$ into domain $\{y - f(x) > 0\}$. (Here, and in the following, inequalities enclosed in braces designate that domain of the phase space where the inequality is fulfilled.) In this case, at point p, the abscissa of the trajectory $\phi(p, t)$ assumes a minimum. If at point p, $z - cx - df(x) < 0$, then trajectory $\phi(p, t)$ intersects surface $y - f(x) = 0$, passing out of domain $\{y - f(x) > 0\}$ into domain $\{y - f(x) < 0\}$. For this case, the abscissa of trajectory $\phi(p, t)$ at point p has a maximum.

Section 4

At this point we formulate a theorem on the behavior of trajectories of system (2.8). To prove this theorem, we need to prove several lemmas.

Lemma 3.1

Let $p \epsilon \{x > 0, z < 0\}$. Then trajectory $\phi(p, t)$ for $t > 0$ intersects surface $x = 0$.

Proof

We assume that $p \epsilon \{x > 0, y - f(x) > 0, z < 0\}$. Then we prove that, with increasing time, trajectory $\phi(p, t)$ of system (2.8) intersects surface $y - f(x) = 0$ and goes into domain $\{x > 0, y - f(x) < 0, z < 0\}$.

Indeed, in domain $\{x > 0,\ y - f(x) > 0,\ z < 0\}$ the x component along all the motions of (2.8) increases, while the y and z components decrease with increasing time. In the domain considered, y on trajectory $\phi(p, t)$ is bounded since it decreases in this domain and is positive. But then, in this domain on trajectory $\phi(p, t)$, x is also bounded.

Indeed, dividing the first equation of system (2.8) by the second, we obtain

$$\frac{dx}{dy} = \frac{y - f(x)}{z - Cx - df(x)} . \tag{3.4}$$

Following from inequalities (2.11) and (2.42), $z - cx - df(x) \leq z_0 < 0$ on trajectory $\phi(p, t)$ for $t > 0$ and, thus, $\phi(p, t) \epsilon \{x > 0,\ y - f(x) > 0,\ z < 0\}$. Therefore, from the boundedness of y in the domain considered, the boundedness of x also follows.

We prove that the z component is also bounded on trajectory $\phi(p, t)$ in domain $\{x > 0,\ y - f(x) > 0,\ z < 0\}$. To verify this, divide the third equation of system (2.8) by the second, thus yielding

$$\frac{dz}{dy} = \frac{-ax + bf(x)}{z - cx - df(x)} . \tag{3.5}$$

Since x is bounded on trajectory $\phi(p, t)$ in the domain considered, and z decreases with increasing time in this domain, because of inequality (2.11), then the right side of equality (3.5) is also bounded. Thus, from the boundedness of y, the boundedness of z also follows. We assume now that trajectory $\phi(p, t)$ remains in domain $\{x > 0,\ y - f(x) > 0,\ z < 0\}$ for all $t \geq 0$. Then $\phi(p, t)$ is bounded for $t \geq 0$. However, in our domain all the coordinates depend monotonically on time; therefore, trajectory $\phi(p, t)$ for $t \to +\infty$ goes to some point of phase space other than the origin. It is known (ref. 26) that such a point can be only an equilibrium position of our system. Yet system (2.8) has only one equilibrium position, considered to be the point (0, 0, 0). Therefore, the assumption that trajectory $\phi(p, t)$ for $t \geq 0$ remains in domain $\{x > 0,\ y - f(x) > 0,\ z < 0\}$, is absurd. Consequently, trajectory $\phi(p, t)$ for $t = t_1 > 0$ intersects the surface $y - f(x) = 0$ and goes into domain $\{x > 0,\ y - f(x) < 0,\ z < 0\}$.

We prove that trajectory $\phi(p, t)$ for all $t > t_1$ cannot lie in domain $\{x > 0,\ y - f(x) < 0,\ z < 0\}$. On the contrary, we assume that $\phi(p, t)$ lies in this domain for all $t > t_1$. We show then that trajectory $\phi(p, t)$ is bounded for $t > t_1$. Indeed, x in this domain decreases and is positive; consequently it is bounded. We assume that the z component along trajectory $\phi(p, t)$ decreases without bound with increasing time. Dividing the third equation of system (2.8) by the first, yields

$$\frac{dz}{dx} = \frac{-ax - bf(x)}{y - f(x)} . \tag{3.6}$$

From this equality, it follows that z on trajectory $\phi(p, t)$ can be unbounded only on the condition that y on $\phi(p, t)$ is bounded for $t > t_1$. But then, as seen from equality (3.5), z is also bounded on $\phi(p, t)$ for $t > t_1$. Consequently, the assumption about the unboundedness of z on trajectory $\phi(p, t)$ for $t > t_1$ is preposterous and means the z component on trajectory $\phi(p, t)$ is bounded for $t > t_1$. From equality

$$\frac{dy}{dx} = \frac{z - cx - df(x)}{y - f(x)}. \qquad (3.7)$$

because of the monotonic decrease of y, and the boundedness of z and x on trajectory $\phi(p, t)$ for $t > t_1$, it follows that y cannot be unbounded. Thus, if trajectory $\phi(p, t)$ for $t > t_1$ remains in domain $\{x > 0, y - f(x) < 0, z < 0\}$, then it is bounded. But all the coordinates along trajectory $\phi(p, t)$ in domain $\{x > 0, y - f(x) < 0, z < 0\}$ vary monotonically. Since z along trajectory $\phi(p, t)$ in this domain decreases with increasing time, then consequently, there is present an equilibrium position belonging to system (2.8) different from $x = y = z = 0$. This, an mentioned above, is not true. But this trajectory cannot intersect plane $z = 0$, since in domain $\{x > 0\}$, z decreases with increasing time along all motions of system (2.8). Trajectory $\phi(p, t)$ also cannot intersect surface $y - f(x) = 0$, for, as stated in the previous paragraph, trajectories of system (2.8) for $z - cx - df(x) < 0$ intersect the surface $y - f(x) = 0$, crossing from domain $\{y - f(x) > 0\}$ into domain $\{y - f(x) < 0\}$, but not for the reverse. Consequently, trajectory $\phi(p, t)$ must intersect surface $x = 0$ for $t = T_p > t_1$. This also proves the lemma.

Remark

Obviously, the assertion of the lemma is true also when $p \epsilon \{x > 0, z = 0\}$ and when $p \epsilon \{x = 0, z \leq 0, y > 0\}$.

The following lemma is proved in a precisely analogous manner.

Lemma 3.2

If $p \epsilon \{x < 0, z > 0\}$, then $\phi(p, t)$, for $t > 0$, intersects plane $x = 0$.

Lemma 3.3

Assume that in system (2.8) the conditions of cases 13 or 17 are not fulfilled. Assume further that $p \epsilon \{x > 0, y - f(x) > 0\}$. Then trajectory $\phi(p, t)$ of system (2.8) for $t > 0$ intersects surface $y = f(x)$ and goes into domain $\{x > 0, y - f(x) < 0\}$.

Proof

As a result of the lemma's assumption, inequality (2.43) is valid. Assume again that $p \epsilon \{x > 0, y - f(x) > 0, z - cx - df(x) > 0\}$. In this domain, x and y increase and z decreases with increasing time t.

Assume the opposite of the assertion of the lemma; i.e., that for all $t > 0$, $\phi(p, t) \epsilon \{x > 0, y - f(x) > 0\}$. We will prove then that $\phi(p, t)$ for $t > 0$ intersects the plane $z - \varkappa x = 0$, where $\varkappa$ is the number calculated in inequality (2.43). Indeed, x and z on trajectory $\phi(p, t)$ in domain $\{x > 0, y - f(x) > 0, z - \varkappa x > 0\}$ are bounded since z in this domain decreases with increasing time. We will prove that the y component in this domain is also bounded on $\phi(p, t)$. To show this, we turn to equality

$$\frac{dy}{dz} = \frac{z - cx - df(x)}{-ax - bf(x)}. \tag{3.8}$$

Following from inequality (2.11), the value of the fraction standing on the right side of this inequality can approach zero only for $x = 0$. Since on $\phi(p, t)$ for $t \geq 0$ in domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x > 0\}$, we see that $x > x(p)$ and, as mentioned above, x on $\phi(p, t)$ in this domain is bounded for $t \geq 0$. We also see that trajectory $\phi(p, t)$ remains in domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x > 0\}$. Therefore, from the boundedness of z, the boundedness of y also follows in this domain. Thus, trajectory $\phi(p, t)$ is bounded for $t \geq 0$ in domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x > 0\}$. Consequently, $\phi(p, t)$ leaves this domain for increasing time. Indeed, assume to the contrary that $\phi(p, t) \epsilon \{x > 0,\ y - f(x) > 0,\ z - \varkappa x > 0\}$ for all $t \geq 0$. But then, because of the boundedness, trajectory $\phi(p, t)$ has ω-limit point q. Since, in the domain considered, x increases along $\phi(p, t)$, it is then clear that $x(q) > 0$. Moreover, resulting from the increase of z along $\phi(p, t)$, $\lim_{t \to +\infty} z(t) = z(q)$. However, from $x(q) > 0$, it follows that for $t > 0$, and sufficiently small, $z(\phi(q, t)) < z(q)$. But then, from the quality of the ω-limit set, a $\tau > 0$ can be found such that $z(\phi(p, \tau)) < z(q)$.

The last inequality contradicts the monotonicity of function $z(\phi(p, t))$ and relation $z(\phi(p, t)) \to z(q)$ for $t \to +\infty$. The contradiction obtained also proves that $\phi(p, t)$ leaves domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x > 0\}$ for $t > 0$. But trajectory $\phi(p, t)$ does not intersect the surface $y - f(x) = 0$ by assumption; consequently, for $t = t_1$, $\phi(p, t)$ intersects plane $z - \varkappa x = 0$ and goes into domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x < 0,\ z > 0\}$. In this domain, y and z are bounded on trajectory $\phi(p, t)$ since they decrease and are positive. We will prove that x is also bounded on $\phi(p, t)$ in this domain. Indeed, resulting from the decreasing of z and the increasing of x, and inequality (2.43),

$$z - cx - df(x) < l_1 < 0, \tag{3.9}$$

is fulfilled in this domain, where l_1 is some constant. Consequently, from the boundedness of y in domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x < 0,\ z > 0\}$ and from equality (3.4), the boundedness of x results in the domain considered. Thus, in our domain, all coordinates on trajectory $\phi(p, t)$ are bounded and monotonically varying with time. Therefore, trajectory $\phi(p, t)$ cannot lie in this domain for all $t > t_1$ and, consequently, for increasing t, leaves it. Trajectory $\phi(p, t)$ cannot intersect plane $z - \varkappa x = 0$ since, in domain $\{x > 0,\ y - f(x) > 0,\ z - \varkappa x < 0\}$, x increases along $\phi(p, t)$, and z decreases with increasing t. Also, $\phi(p, t)$ cannot intersect plane $z = 0$ since x increases for $y > f(x)$. Therefore, for $t = t_2 > t_1$, trajectory $\phi(p, t)$ intersects either plane $z = 0$ or surface $y - f(x) = 0$. In the second case, the lemma is proved. If, however, $\phi(p, t)$ for $t = t_2$ intersects plane $z = 0$, then, in this case, it crosses into domain $\{x > 0,\ y - f(x) > 0,\ z < 0\}$. But then, as shown in the proof of lemma 3.1, for $t = t_3 > t_2$, trajectory $\phi(p, t)$ intersects surface $y - f(x) = 0$ and goes into domain $\{x > 0,\ y - f(x) < 0\}$. This also concludes the proof of lemma 3.3.

Remark

Obviously, the assertion of the lemma is true also when $p \epsilon \{x \geq 0,\ y - f(x) \geq 0,\ z - cx - df(x) > 0\}$.

Lemma 3.4

Assume that in system (2.8) either condition 13 or 17 is not fulfilled. Assume further that $p \epsilon \{x < 0,\ y - f(x) < 0\}$. Then trajectory $\phi(p, t)$ for $t > 0$ intersects surface $y - f(x) = 0$ and goes into domain $\{x < 0,\ y - f(x) > 0\}$. The proof of this lemma is analogous to the proof of lemma 3.3.

Lemma 3.5

If for all $t \geq 0$, $\phi(p, t)$ is contained in $\epsilon\{x > 0, y - f(x) \leq 0, z > 0\}$, then $\phi(p, t)$ goes to the origin as $t \to \infty$.

Proof

Since, along $\phi(p, t)$, the x and z components decrease with increasing time, they are bounded on this trajectory for $t \geq 0$.

Assume that

$$g_1 = \max f(x) \text{ for } 0 \leq x \leq x(p). \tag{3.10}$$

Then for $t \geq 0$, trajectory $\phi(p, t)$ goes into domain $\{x > 0, y - f(x) \leq 0, z > 0\}$, and, therefore, the y component is bounded on it above g_1. Besides, from equality (3.7), from the boundedness of x and z and from the monotonic decrease of x along trajectory $\phi(p, t)$ with increasing time, we see that the y component is bounded on $\phi(p, t)$ for $t \geq 0$ also from below. Thus, trajectory $\phi(p, t)$ is positively stable in the sense of LaGrange and, as a consequence, has an ω-limit point q with coordinates x_0, y_0, z_0.

We will prove that $x_0 = y_0 = z_0 = 0$. Indeed, since for $t \geq 0$, z decreases monotonically along trajectory $\phi(p, t)$ with increasing time, there must be fulfilled the relations

$$z(\phi(p, t)) \geq z_0 \text{ for } t \geq 0 \tag{3.11}$$

and

$$\lim_{t\to\infty} z(\phi(p, t)) = z_0. \tag{3.12}$$

Point q is the ω-limit for trajectory $\phi(p, t)$, lying inside or on the boundary of that domain in which trajectory $\phi(p, t)$ lies for $t \geq 0$, and, consequently, $q\epsilon\{x \geq 0, y - f(x) \leq 0, z \geq 0\}$.

Since $\phi(q, t)$ is an ω-limit trajectory for $\phi(p, t)$, then, for all t,

$$\phi(q, t)\epsilon\{x \geq 0, y - f(x) \leq 0, z \geq 0\}.$$

We assume now that point q does not coincide with the equilibrium position (0, 0, 0) of system (2.8). From the preceding reasoning, it is easy to see that for any $t_1 > 0$

$$z(\phi(q, t_1)) < z_0. \tag{3.13}$$

Since $\phi(q, t)$ is an ω-limit trajectory for $\phi(p, t)$, as a consequence of inequality (3.13) there exists a $\tau > 0$ for which there is fulfilled the inequality

$$z(\phi(p, \tau)) < z_0. \tag{3.14}$$

The last inequality contradicts inequality (3.11). The contradiction obtained also proves that point q coincides with the origin. As a result, trajectory $\phi(p, t)$ has point $x = y = z = 0$ as its unique ω-limit point. Thus, the lemma is proved.

Lemma 3.6

If for $t \geq 0$

$$\phi(p, t) \epsilon \{x < 0, y - f(x) \geq 0, z < 0\},$$

then $\phi(p, t)$ goes to the origin for $t \to +\infty$. The proof is analogous to the proof of lemma 3.5.

Theorem 3.1

Assume that in system (2.8) conditions of either case 13 or 17 are not fulfilled; then any positive half-trajectory of system (2.8), wholly lying in one of the half-spaces $x \geq 0$ or $x \leq 0$, goes to the origin.

Proof

Assume for definiteness that for $t \geq 0$ the relation $\phi(p, t) \epsilon \{x \geq 0\}$ is fulfilled. Resulting from lemma 3.1, $\phi(p, t) \epsilon \{x \geq 0, z > 0\}$ (provided p does not coincide with the point (0, 0, 0), which we will also assume for the proof of this theorem). Two cases are possible: either such a T exists that for $t > T$, $\phi(p, t) \epsilon \{x > 0, y - f(x) \leq 0, z > 0\}$, or such a T does not exist. In the first case, the proof is concluded by a simple reference to lemma 3.5. Turning to the second case, by virtue of lemma 3.3, there can be found a

$$\theta_1 > 0 \text{ such that } \phi(p, \theta_1) \epsilon \{x > 0, y - f(x) < 0, z > 0\}.$$

Since, from the suppositions it cannot occur that $\phi(p, t) \epsilon \{x > 0, y - f(x) < 0, z > 0\}$ for all $t \geq \theta_1$, then a $\tau_1 > \theta_1$ can be found such that on trajectory $\phi(p, t)$

$$y(\tau_1) = f(x(\tau_1)). \tag{3.15}$$

From the reasoning in section 3, it is clear that for this

$$z(\tau_1) - \varkappa x(\tau_1) \geq 0 \tag{3.16}$$

on trajectory $\phi(p, t)$.

According to lemma 3.5, we will be able to choose such a number $\theta_2 > \tau_1 + 1$ that $\phi(p, \theta_2) \epsilon \{x > 0, y - f(x) < 0, z > 0\}$. For this θ_2, just as above, a $\tau_2 > \theta_2$ can be found so that

$$y(\tau_2) = f(x(\tau_2)),\ z(\tau_2) - \varkappa\, x(\tau_2) \geq 0. \tag{3.17}$$

Continuing this process further, we can choose a sequence $\tau_1, \tau_2, \ldots$ of instants of time such that $\tau_k > \tau_{k-1} + 1$ and

$$y(\tau_k) = f(x(\tau_k)), \tag{3.18}$$

$$z(\tau_k) - \varkappa\, x(\tau_k) \geq 0 \tag{3.19}$$

on trajectory $\phi(p, t)$.

Since trajectory $\phi(p, t)$ for $t \geq 0$ lies in the half-space $x \geq 0$, then along this trajectory, z decreases with increasing time for $t \geq 0$, as follows from condition (2.14). Therefore, we can write

$$z(\tau_k) \leq z(p) \tag{3.20}$$

on trajectory $\phi(p, t)$. Also from (3.19), we obtain

$$x(\tau_k) \leqslant \frac{1}{\varkappa} z(p) \tag{3.21}$$

on $\phi(p, t)$. Assume now that

$$g_2 = \max f(x) \quad \text{for} \quad 0 \leqslant x \leqslant \frac{1}{\varkappa}\, z(p). \tag{3.22}$$

From (3.18), (2.10), (3.21) and (3.22), there results the following inequality:

$$0 \leq y(\phi(p, \tau_k)) \leq g_2. \tag{3.23}$$

From inequalities (3.21), (3.22) and (3.23), the sequence of points $\phi(p, \tau_k)$ is bounded, and, consequently, it has a limit point q with coordinates x_0, y_0, z_0.

Since $\tau_k > \tau_{k-1} + 1$, $\lim\limits_{k\to\infty} \tau_k = \infty$, and, consequently, point q is an ω-limit point for trajectory $\phi(p, t)$. We will prove that point q coincides with the origin. Indeed, along trajectory $\phi(p, t)$, z monotonically decreases with increasing time and, therefore, there must be fulfilled by the relations

$$z(\phi(p, t)) > z_0 \text{ for } t \geq 0 \tag{3.24}$$

and

$$\lim_{t\to\infty} z(\phi(p, t)) = z_0. \tag{3.25}$$

Trajectory $\phi(q, t)$ is an ω-limit for $\phi(p, t)$. Therefore, for all t, $\phi(q, t)\epsilon\{x \geq 0,\ z \geq 0\}$. Assume that q does not coincide with the origin; then, obviously, a $t_1 > 0$ can be found for which

$$z(\phi(q, t_1) < z_0. \tag{3.26}$$

Since point $\phi(q, t_1)$ is an ω-limit for trajectory $\phi(p, t)$, an instant of time $t_2 > 0$ can be found such that

$$z(\phi(p, t_2) < z_0. \tag{3.27}$$

This inequality contradicts inequality (3.24). The contradiction obtained also proves that point q coincides with the origin. Analogously, it is proved that any other ω-limit point of trajectory $\phi(p, t)$ coincides with the origin. As a consequence, $\phi(p, t)$ goes to this equilibrium position for $t \to +\infty$. Thus, the theorem is proved.

Section 5

In this section, we will investigate system (2.15) where parameters a and b are subjected to the conditions of cases 1 and 4; i.e., we will consider that the relations $0 \leq b < 1$ and $a > 0$ are fulfilled. Assume $c = \frac{a}{1 - b}$. From inequalities (2.20) and (2.23) there must follow

$$\frac{f(x)}{x} > c \text{ for } x \neq 0. \tag{3.28}$$

Assume that

$$f(x) = cx + \alpha(x). \tag{3.29}$$

Then inequality (3.28) has the form

$$\frac{\alpha(x)}{x} > 0 \text{ for } x \neq 0. \tag{3.30}$$

The following lemma is true.

Lemma 3.7

Assume that the inequalities are fulfilled: $0 \leq b < 1$, $a > 0$, and $c \geq 1$. Let $p \epsilon \{x \geq 0,\ y = f(x) > 0\}$. Then trajectory $\phi(p, t)$ for $t > 0$ intersects plane $x = 0$.

Proof

If $z(p) \leq 0$, then the assertion of this lemma follows directly from lemma 3.1. Let $z(p) > 0$. Assume to begin with that $p \epsilon \{x \geq 0,\ y - f(x) > 0,\ z - x > 0\}$. As was proved for lemma 3.3, trajectory $\phi(p, t)$ intersects plane $z - x = 0$ for $t = t_1 > 0$, whereupon inequality (3.9) will be fulfilled on this trajectory. Thus, if $p \epsilon \{x \geq 0, y - f(x) > 0, z - x > 0\}$, a $t_1 \geq 0$ exists such that on $\phi(p, t)$ the following relations are fulfilled:

$$z(t_1) - x(t_1) \leq 0, \tag{3.31}$$

$$y(t_1) - f(x(t_1)) \geq 0. \tag{3.32}$$

But if $p \epsilon \{x > 0, y - f(x) > 0, z - x \leq 0\}$, then assume $t_1 = 0$; thus relations (3.31) and (3.32) on trajectory $\phi(p, t)$ will also be fulfilled. Because of inequality (3.28), on $\phi(p, t)$ there is fulfilled the inequality

$$y(t_1) \geq f(x(t_1)) > cx(t_1). \tag{3.33}$$

Assume that

$$f(x(t_1)) - cx(t_1) = h > 0.$$

Then from (3.33) we obtain

$$y(t_1) \geq cx(t_1) + h \tag{3.34}$$

on trajectory $\phi(p, t)$. Returning now to equality (3.8), we will rewrite it in the form

$$\frac{dy}{dz} = \frac{x}{ax + bf(x)} - \frac{z}{ax + bf(x)}.$$

Therefore, from relation (3.30) and the definition of the number c,

$$\frac{dy}{dx} = \frac{x}{cx + b\alpha(x)} - \frac{z}{Cx + b\alpha(x)}.$$

From conditions $b \geq 0$, $c \geq 1$ of the last equality, the relation

$$\frac{dy}{dz} < 1 \tag{3.35}$$

is valid for $x > 0$ and $z > 0$. Let $T > t_1$ be an arbitrary number so that for $t \epsilon (t_1, T)$ on trajectory $\phi(p, t)$, $x > 0$, $z > 0$ results. Then on trajectory $\phi(p, t)$ for $t \epsilon (t_1, T)$, inequality (3.35) is fulfilled. By integrating this inequality along trajectory $\phi(p, t)$ from t_1 to T, we obtain

$$y(\phi(p, T)) - y(\phi(p, t_1)) > z(\phi(p, T)) - z(\phi(p, t_1))$$

or

$$y(\phi(p, T)) > z(\phi(p, T)) + y(\phi(p, t_1)) - z(\phi(p, t_1)).$$

Therefore, from relations (3.31), (3.34) and $c \geq 1$ we obtain

$$y(\phi(p,\ T)) > z(\phi(p,\ T)) + h. \tag{3.36}$$

From this relation the inequality

$$x(\phi(p,\ T)) > 0 \tag{3.37}$$

must be fulfilled since for $y > 0$ the trajectories of system (2.15) intersect the plane $x = 0$, passing from the half-space $x < 0$, into the half-space $x > 0$.

Now we will prove that $\phi(p, t)$ intersects the plane $z = 0$ for $t > t_1$. Indeed, we assert to the contrary that this is not so; i.e., we assume that for $t > t_1$, $z(\phi(p, t)) > 0$. However, (3.37) causes $x(\phi(p, t)) > 0$ for all $t > t_1$ (since for T we can then choose any number larger than t_1). From theorem 3.1 trajectory $\phi(p, t)$ goes to the origin for $t \to \infty$ in this case, which is impossible because of relations (3.36) and $z(\phi(p, t)) > 0$ for $t > t_1$. The resulting contradiction proves that $\phi(p, t)$ intersects plane $z = 0$ for $t = t_2 > t_1$. For this it is clear that $x(\phi(p, t)) > 0$ for $t \epsilon \{t_1, t_2\}$. From lemma 3.1, it can be asserted that trajectory $\phi(p, t)$ intersects plane $x = 0$. Thus, the lemma is proved.

Assume that $0 \leq b < 1$, $a > 0$, $c \geq 1$ and $p \epsilon \{x \geq 0,\ y - f(x) > 0\}$. Then, because of lemma 3.7, trajectory $\phi(p, t)$ intersects plane $x = 0$ for $t > 0$. Let $t_p > 0$ be the first instant after $t = 0$ that trajectory $\phi(p, t)$ intersects the plane $x = 0$. Then from the same proof as for lemma 3.7, it is seen that on trajectory $\phi(p, t)$ the relations

$$y(t_p) < 0,\ z(t_p) < 0. \tag{3.38}$$

are fulfilled. Lemma 3.8 is proved analogously.

Lemma 3.8

If $0 \leq b < 1$, $a > 0$, $c \geq 1$ and $p \epsilon \{x \leq 0,\ y - f(x) < 0\}$, trajectory $\phi(p, t)$ intersects plane $x = 0$ for $t > 0$. From lemmas 3.7 and 3.8, the following theorems are consequences.

Theorem 3.2

Let the inequalities $0 \leq b < 1$, $a > 0$ and $c \geq 1$ be fulfilled and let point p lie on plane $x = 0$. Thus, trajectory $\phi(p, t)$ of system (2.15) intersects plane $x = 0$ for $t > 0$.

Theorem 3.3

Assume that inequalities $0 \leq b < 1$, $a > 0$, $c \geq 1$ are fulfilled. Let point p, different from point $x = y = z = 0$, lie on plane $x = 0$, let $t_1 > 0$ be the first instant after $t = 0$ that trajectory $\phi(p, t)$ intersects plane

$x = 0$, and let $t_2 > t_1$ be the first instant after t_1 of the intersection of $\phi(p, t)$ with plane $x = 0$. Then one of two things is possible:

(1) either

$$y(\phi(p, t_1)) > 0, \; z(\phi(p, t_1)) > 0$$

and then

$$y(\phi(p, t_2)) < 0, \; z(\phi(p, t_2)) < 0,$$

(2) or

$$y(\phi(p, t_1)) < 0, \; z(\phi(p, t_1)) < 0$$

and then

$$y(\phi(p, t_2)) > 0, \; z(\phi(p, t_2)) > 0.$$

Section 6

In this section we will formulate a lemma of later importance. Consider two trajectories of system (2.8); $\phi(p_1, t)$ and $\phi(p_2, t)$. Let the initial points p_1 and p_2 have coordinates $x_1^{(o)}$, $y_1^{(o)}$, $z_1^{(o)}$ and $x_2^{(o)}$, $y_2^{(o)}$, $z_2^{(o)}$ correspondingly. Assume that the relations

$$y_2^{(0)} \geqslant f(x_2^{(0)}), \; z_2^{(0)} > cx_2^{(0)} + df(x_2^{(0)}), \; x_1^{(0)} = x_2^{(0)} \geqslant 0, \tag{3.39}$$

$$z_1^{(0)} \geqslant z_2^{(0)}, \; y_1^{(0)} \geqslant y_2^{(0)}. \tag{3.40}$$

are fulfilled. For this it is assumed that one of the inequalities in (3.40) is fulfilled in the strict sense. From (3.39) and (3.40) follow the inequalities

$$y_1^{(0)} \geqslant f(x_1^{(0)}), \; z_1^{(0)} > cx_1^{(0)} + df(x_1^{(0)}). \tag{3.41}$$

We will consider trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ on those segments from points p_1 and p_2 to the first intersections with the surface $y = f(x)$ after $t = 0$ (and if such does not occur, then to infinity). On this section, the x components of trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ are monotonic functions of time t. Therefore, the y and z components of trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ are single-valued, continuous functions of the x components on the part considered. We will write y and z on trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ as functions of x for which we will provide the coordinates of trajectory $\phi(p_1, t)$ with index 1 and coordinates of trajectory $\phi(p_2, t)$ with index 2. The following lemma is true.

Lemma 3.9

For trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ the inequalities

$$y_1(x) > y_2(x), \tag{3.42}$$

$$z_1(x) > z_2(x), \tag{3.43}$$

occur for all those $x > x_1^{(o)}$ for which both trajectories belong to the segments considered.

Proof

We take arbitrary $x_1 > x_1^{(o)}$, such that for $x \epsilon (x_1^{(o)}, x_1]$, trajectories $\phi(p_1, t)$ and $\phi(p_2, t)$ do not intersect surface $y - f(x) = 0$, and we will prove that inequalities (3.42) and (3.43) are fulfilled for all $x \epsilon (x_1^{(o)}, x_1]$. By this, the lemma will also be proved.

From relation (3.40), both inequalities (3.42) and (3.43) are fulfilled for $x \epsilon (x_1^{(o)}, x_1^{(o)} + \delta]$, where δ is a sufficiently small number. Indeed, by virtue of continuity, they are fulfilled also for $x \epsilon [x_1^{(o)}, x_1^{(o)} + \delta]$ from inequalities (3.42) and (3.43) which are strictly fulfilled for $x = x_1^{(o)}$ (and such exists because of the assumption). But then, because of equations (3.6) and (3.7), the second of these inequalities is also fulfilled for $x \epsilon (x_1^{(o)}, x_1^{(o)} + \delta]$.

Assume now, in spite of the lemma's assertion such an $x^* \epsilon [x_1^{(o)} + \delta, x_1]$ exists, that for $x = x^*$, either inequality (3.42) or inequality (3.43) is violated, and for $x \epsilon (x_1^{(o)} + \delta, x^*)$ both are fulfilled; i.e., x^* is the first point at which one of inequalities (3.42) and (3.43) is violated. However, inequality (3.43) cannot be violated for $x = x^*$ since

$$\frac{dz_1}{dx} > \frac{dz_2}{dx} \quad \text{for} \quad x \in (x_1^{(0)} + \delta, x^*), \tag{3.44}$$

This follows from inequality (3.42) which, by assumption, is fulfilled for $x \epsilon (x_1^{(o)} + \delta, x^*)$.

Inequality (3.42) also is not violated for $x = x^*$. Indeed, assume

$$y_1(x^*) = y_2(x^*). \tag{3.45}$$

Then, since inequality (3.43) is fulfilled for $x \epsilon (x_1^{(o)} + \delta, x^*)$, it must be fulfilled also for $x = x^*$ by virtue of the theorem on uniqueness; in this case, resulting from (3.7), we have

$$\left.\frac{dy_1}{dx}\right|_{x-x^*} > \left.\frac{dy_2}{dx}\right|_{x-x^*} .$$

Because of the continuity, the last inequality is fulfilled also for x, sufficiently close to x^*, but less than it. This contradicts equality (3.45), and the contradiction obtained also proves the lemma.

Chapter IV. ON THE GLOBAL STABILITY OF MOTION

In this chapter, we will give several sufficient conditions for the global stability of the null solution of system (2.8). To do this, in many cases we will construct a Lyapunov function for the system in the form "integral of the nonlinearity plus a quadratic form of the coordinate of the phase space." Many authors (refs. 9–16) have constructed Lyapunov functions in this way for systems of the Ayzerman type in special cases.

Section 7

This section will show one very simple example which in some cases enables a Lyapunov function to be constructed for system (2.8) of the special type pointed out above. Yet in some other cases it proves the absence of Lyapunov functions of such a type. We will prove the following.

Lemma 4.1

If the quadratic form

$$v = W(x_1, x_2, \ldots, x_n) + \frac{\mu}{2} F x_k^2, \tag{4.1}$$

where F is a constant number and W is a quadratic form of the variables $x_1, x_2, \ldots, x_n$ with coefficients independent of F, is positive definite for any $F \epsilon (\gamma, \delta)$, then the function

$$v_1 = W(x_1, x_2, \ldots, x_n) + \mu \int_0^{x_k} f(x)\, dx \tag{4.2}$$

is also positive definite for any continuous f(x) satisfying the conditions

$$f(0) = 0, \ \gamma < \frac{f(x)}{x} < \delta \quad \text{for} \quad x \neq 0. \tag{4.3}$$

Proof

Choose an arbitrary continuous function f(x) satisfying the conditions of (4.3). We will prove that at any point $(x_{10}, z_{20}, \ldots, x_{n0}) \neq (0, 0, \ldots, 0)$, function (4.2) is positive; the lemma is also proved by this. If $x_{k0} = 0$, then

$$v(x_{10}, x_{20}, \ldots, x_{n0}) = v_1(x_{10}, x_{20}, \ldots, x_{n0}) = W(x_{10}, x_{20}, \ldots, x_{n0}),$$

and our assertion is proved. Let $x_{k0} \neq 0$, and multiply inequality (4.3) by x and integrate from 0 to x_{k0}, thus obtaining

$$\frac{1}{2}\gamma x_{k0}^2 < \int_0^{x_{k0}} f(x)\,dx < \frac{1}{2}\,\delta x_{k0}^2. \tag{4.4}$$

Assume that

$$F_0 = \frac{2}{x_{k0}^2}\int_0^{x_{k0}} f(x)\,dx. \tag{4.5}$$

From (4.4) it follows that $F_0 \epsilon (\gamma, \delta)$, but then from hypothesis

$$v(x_{10}, x_{20}, \ldots, x_{n0}) = W(x_{10}, x_{20}, \ldots, x_{n0}) + \frac{\mu}{2} F_0 x_{k0}^2 < 0.$$

Therefore, resulting from (4.5),

$$v_1(x_{10}, x_{20}, \ldots, x_{n0}) > 0.$$

The lemma is proved.

Suppose now that in functions v and v_1, the quadratic form W has coefficients dependent only on the coefficients a_{ij} of systems (1) and (2). Designate by $\dot{v}$ and $\dot{v}_1$ the derivative on time of the functions v and v_1 taken because of the differential equations of systems (1) and (2) respectively. The following lemma is true.

Lemma 4.2

If for every $F\epsilon(\gamma, \delta)$, $\dot{v} \leq 0$, then for every continuous f(x) satisfying conditions (4.3), $\dot{v}_1 \leq 0$.

Proof

We will take an arbitrary continuous function f(x) satisfying conditions (4.3) and show that by the hypotheses of the lemma, $\dot{v}_1 \leq 0$ at any fixed point $(x_{10}, x_{20}, \ldots, x_{n0})$ of the phase space: this proves the lemma. If $x_{k0} = 0$, then the lemma proceeds from the fact that $f(x_{k0}) = Fx_{k0} = 0$ and from the forms $\dot{v}$ and $\dot{v}_1$. But let $x_{k0} \neq 0$, and then assume $F_0 = \dfrac{f(x_{k0})}{x_{k0}}$.

From conditions (4.3) it follows that $a \in F_0 \epsilon(\gamma, \delta)$, but then $\dot{v} \leq 0$ for $x_i = x_{i0} (i = 1, 2, \ldots, n)$. Therefore, the assertion of the lemma also results from equality $f(x_{k0}) = F_0 x_{k0}$ and from the forms $\dot{v}$ and $\dot{v}_1$.

From lemmas 4.1 and 4.2, it follows that if there exists for system (1) a Lyapunov function of type (4.1) for any $F \epsilon(\gamma, \delta)$, then for system (2) there also exists a Lyapunov function for any continuous f(x) satisfying conditions (4.3), and this function has the form (4.2).

Further, it is clear that if it is impossible to choose for system (1) a function v of type (4.1) having a negative derivative $\dot{v}$ for all $F \epsilon(\gamma, \delta)$, then also for system (2) there exists no Lyapunov function of the type "integral of the nonlinearity plus a quadratic form of the coordinates of the phase space."

Thus, the question of the existence and the construction for system (2) of a Lyapunov function of the special type chosen above is seen to be a question of the existence and construction for system (1) of a Lyapunov function of type (4.1).

Section 8

This section will investigate cases 1, 4, 5, and 7 introduced in chapter II. Assume as previously that

$$c \doteq \frac{a}{1-b}. \tag{4.6}$$

System (2.15) then takes on the following form:

$$\dot{x} = y - cx - \alpha(x), \quad \dot{y} = z - x, \quad \dot{z} = -cx - b\alpha(x), \tag{4.7}$$

where $\alpha(x)$ was introduced by equality (3.29). Function $\alpha(x)$ obeys the following inequalities:

In cases 1, and 4 (i.e., for $0 \leq b < 1$, $a = 0$),

$$\frac{\alpha(x)}{x} > 0 \quad \text{for} \quad x \neq 0; \tag{4.8}$$

In case 5 (i.e., for $b < 0$, $a > 0$),

$$0 < \frac{\alpha(x)}{x} < -\frac{c}{b} \quad \text{for} \quad x \neq 0; \tag{4.9}$$

In case 7 (i.e., for $b > 1$, $a < 0$),

$$-\frac{c}{b} < \frac{\alpha(x)}{x} < 0 \quad \text{for} \quad x \neq 0. \tag{4.10}$$

In these cases for system (4.7), we start to look for a Lyapunov function of the type "integral of the nonlinearity plus a quadratic form in the coordinates of the phase space." For this case, including system (4.7), consider the system

$$\dot{x} = y - cx - Ax, \quad \dot{y} = z - x, \quad \dot{z} = - cx - bAx, \tag{4.11}$$

where the quantity A obeys the same inequality that the expression $\alpha(x)/x$ depends on in the case considered.

Substitute in systems (4.7) and (4.11) the following change of variables:

$$x_1 = c^2x - cy + z, \; y_1 = z - x, \; z_1 = y, \tag{4.12}$$

$$x = \frac{x_1 - y_1 + cz_1}{c^2 + 1}.$$

Then systems (4.7) and (4.11) will have the forms

$$\begin{gathered} \frac{dx_1}{dt} = - cx_1 - (c^2 + b)\,\alpha(x), \\ \frac{dy_1}{dt} = - z_1 + (1 - b)\,\alpha(x), \quad \frac{dz_1}{dt} = y_1 \end{gathered} \tag{4.13}$$

and

$$\begin{gathered} \frac{dx_1}{dt} = - cx_1 - (c^2 + b)\,Ax, \\ \frac{dy_1}{dt} = - z_1 + (1 - b)\,Ax, \quad \frac{dz_1}{dt} = y_1. \end{gathered} \tag{4.14}$$

For system (4.14), look for a Lyapunov function in the form

$$\begin{aligned} v = \frac{1}{2}\,b_{11}x_1^2 + \frac{1}{2}\,b_{22}y_1^2 + \frac{1}{2}\,b_{33}z_1^2 + b_{12}\,x_1y_1 \\ + b_{13}x_1z_1 + b_{23}y_1z_1 + \frac{1}{2}\,\mu Ax^2, \end{aligned} \tag{4.15}$$

where the numbers b_{ik} and μ are, for the present, not defined. We require that the derivatives of this function on time, taken by virtue of the differential equations of system (4.14) for any A satisfying suitable inequalities, be nonpositive:

$$\begin{aligned} \dot{v} = & - b_{11}cx_1^2 - b_{11}(c^2 + b)\,Axx_1 - b_{22}y_1z_1 \\ & + b_{22}(1 - b)\,Axy_1 + b_{33}y_1z_1 - b_{12}cx_1y_1 \\ & - b_{12}(c^2 + b)\,Axy_1 - b_{12}x_1z_1 + b_{12}(1 - b)\,Axx_1 \\ & - b_{13}cx_1z_1 - b_{13}(c^2 + b)Axz_1 + b_{13}x_1y_1 - b_{23}z_1^2 \\ & + b_{23}(1 - b)\,Axz_1 + b_{23}y_1^2 + \mu Ax\,(y - cx - Ax). \end{aligned} \tag{4.16}$$

Since $\dot{v}$ must be negative definite for any A satisfying appropriate inequalities, then obviously the function v must be negative definite also for A = 0. From (4.16), it then follows that

$$b_{23} = 0, \quad b_{22} = b_{33}, \tag{4.17}$$

$$-b_{12}c + b_{13} = 0, \quad -b_{12} - b_{13}c = 0. \tag{4.18}$$

From equality (4.18), it follows that

$$b_{12} = b_{13} = 0. \tag{4.19}$$

From (4.15), (4.16), (4.17) and (4.19), we obtain

$$v = \frac{1}{2} b_{11}x_1^2 + \frac{1}{2} b_{22}y_1^2 + \frac{1}{2} b_{22}z_1^2 + \frac{1}{2} \mu A x^2 \tag{4.20}$$

and

$$\begin{aligned} \dot{v} = &- b_{11}cx_1^2 - b_{11}(c^2 + b) A x x_1 + b_{22}(1 - b) A x y_1 \\ &+ \frac{\mu A x}{c^2 + 1} [- c x_1 + c y_1 + z_1 - (c^2 + 1) A x]. \end{aligned} \tag{4.21}$$

From the conditions of the cases considered, it is seen that $c > 0$. Therefore, following from (4.21), it must be true that $b_{11} \geq 0$. For $b_{11} = 0$ and a sufficiently small $|A|$, function $\dot{v}$, as it is not difficult to prove, is the sign variable; consequently, it must be true that $b_{11} > 0$. Since function v is of interest to us only to within a positive constant multiplying factor, then it can be taken that $b_{11} = 1$. Then the function $\dot{v}$ can be written in the form

$$\begin{aligned} \dot{v} = &- c x_1^2 - \left[(c^2 + b) + \frac{c\mu}{c^2 + 1}\right] A x x_1 \\ &+ \left[b_{22}(1 - b) + \frac{\mu c}{c^2 + 1}\right] A x y_1 + \frac{\mu}{1 + c^2} x z_1 - \mu A^2 x^2. \end{aligned} \tag{4.22}$$

For $x_1 = 0$, the coefficients for Ax must be proportional to x since in the opposite case for small A, function $\dot{v}$ will change sign. Therefore,

$$b_{22}c(1 - b) + \frac{\mu c^2}{1 + c^2} = -\frac{\mu}{1 + c^2}.$$

Thus, we obtain:

$$b_{22} = -\frac{\mu}{c(1 - b)}. \tag{4.23}$$

Substituting the found value of b_{22} into $\dot{v}$ and changing $x_1 - y_1 + cz_1$ to $(c^2 + 1)x$, we obtain

$$\dot{v} = -cx_1^2 - \left[(c^2 + b) + \frac{\mu}{c}\right] Axx_1 + \frac{\mu}{c} Ax^2 - \mu A^2 x^2. \tag{4.24}$$

The condition for negativeness of the last function consists of

$$-\mu A + \mu c A^2 \geqslant \frac{1}{4}\left[(c^2 + b) + \frac{\mu}{c}\right]^2 A^2. \tag{4.25}$$

Consider now case 7; i.e., the case when $b > 1$, $a < 0$. According to (4.10), in this case

$$-\frac{c}{b} < A < 0. \tag{4.26}$$

Cancelling A in inequality (4.25) for $A < 0$, we obtain

$$-\mu + \mu c A \leqslant \frac{1}{4}\left[(c^2 + b) + \frac{\mu}{c}\right]^2 A. \tag{4.27}$$

Inequality (4.27) is strictly fulfilled for $A = 0$ and for any $\mu > 0$. Therefore, if we choose $\mu > 0$, then inequality (4.27) will be fulfilled for sufficiently small $|A|$. We now seek to choose $\mu > 0$ such that for $A = -c/b$ inequality (4.27) will also be fulfilled. If such a μ is found, then inequality (4.27) will be fulfilled for any A satisfying inequality (4.26) since in (4.27) A enters linearly. Thus, assuming in (4.27) that $A = -c/b$, we obtain

$$-4\frac{b + c^3}{b}\mu \leqslant -\frac{c}{b}\left[(c^3 + b)^3 + \frac{2\mu(c^3 + b)}{c} + \frac{\mu^2}{c^2}\right]$$

or

$$-\frac{c}{b}\left[(c^2 + b) - \frac{\mu}{c}\right]^2 \geqslant 0. \tag{4.28}$$

Since $c > 0$ and $b > 0$, the preceding inequality must be fulfilled only for the following condition:

$$\mu = c(c^2 + b). \tag{4.29}$$

It is obvious that μ chosen in this way is positive, and, as a result, inequality (4.27) for such μ is fulfilled for any $A \epsilon (-c/b, 0)$, as is not difficult to see in the strict sense. Then for any x and x_1 satisfying inequality $x_1^2 + x^2 > 0$ and for any $A \epsilon (-c/b, 0)$, we see that $\dot{v} < 0$.

Thus, in case 7 for system (2.15) there exists a Lyapunov function of the type

$$v_1 = \frac{1}{2}x_1^2 + \frac{1}{2}\frac{c^3 + b}{b - 1}y_1^2 + \frac{1}{2}\frac{c^2 + b}{b - 1}z_1^2 + c(c^3 + b)\int_0^x \alpha(x)\,dx, \tag{4.30}$$

where the variables x_1, y_1, and z_1 come from formula (4.12). Derivative $\dot{v}_1$ of function v_1 on time, taken by virtue of the differential equations of system (2.15), will be negative definite as a consequence of lemma (4.2). From the proof of lemma (4.2) it follows that $\dot{v}_1 < 0$ for $x^2 + x_1^2 > 0$. Now it is easy to prove the following assertion.

Theorem 4.1

In case 7 for $b > 1$, $\alpha < 0$, the null solution of system (2.15) is *globally stable* for any function $f(x)$ satisfying the generalized Hurwitz conditions of (2.16) and (2.26).

Proof

We will prove first that function v_1 of equality (4.30) is positive definite. This is certainly not difficult to achieve with the assistance of Sylvester's criterion and lemma 4.1. However, we will take advantage of another way. Suppose that at a certain point $p \neq (0, 0, 0)$ of the phase space, $v_1(p) \leq 0$. From the type of the function v_1, it follows that $x(p) \neq 0$. But then, as proven above, $\dot{v}_1(p) < 0$. Consequently, for all $t > 0$ on path $\phi(p, t)$ of system (2.15), the result is $v_1 < 0$. But then path $\phi(p, t)$ cannot intersect plane $x = 0$ since on this plane, $v_1 \geq 0$. Moreover, $\phi(p, t)$ cannot go to the origin since $v_1(0, 0, 0) = 0$. Thus we obtain a contradiction with theorem 3.1. This contradiction proves that function v_1 is positive definite. Thus, as a consequence of the fact that $\dot{v} \leq 0$, the null solution of system (2.15) in the case investigated is stable in the sense of Lyapunov.

We prove now that all the conditions of theorem 1.1 are fulfilled. For the hyperplane L figuring in condition 3 of this theorem, choose $x = 0$; then because of theorem 3.1, condition 3a will be fulfilled. Instead of function v figuring in condition 3b of theorem 1.1, we take the function

$$v_1^* = v_1(0,\ y,\ z) = \frac{1}{2}(z - cy)^2 + \frac{1}{2}\frac{c^2+b}{b-1}y^2 + \frac{1}{2}\frac{c^2+b}{b-1}z^2,$$

i.e., that function from the substitution of $x = 0$ into v_1. Condition 3b of theorem 1.1 will be fulfilled in this case. As mentioned above, $x \neq 0$ for $\dot{v}_1 < 0$. But from the reasoning in section 3, it follows that any path $\phi(p, t)$ of system (2.15) in which $p \neq (0, 0, 0)$ *will intersect plane* $x = 0$ *only at isolated points. Therefore*, it follows that the last condition of theorem 1.1 in our case is also fulfilled, and a reference to this theorem thus completes the proof.

Consider now case 5 for $b < 0$, $\alpha > 0$. According to (4.9), in this case there is necessarily fulfilled the inequality

$$0 < A < -\frac{c}{b}. \tag{4.31}$$

Dividing inequality (4.25) by $A > 0$, we obtain

$$-\mu + \mu c A \geqslant \frac{1}{4}\left[(c^2 + b) + \frac{\mu}{c}\right]^2 A. \tag{4.32}$$

In the case considered, inequality (4.32) is fulfilled for sufficiently small A if $\mu < 0$. Therefore, it is required to find a $\mu < 0$ such that for $A = -c/b$, inequality (4.32) would be fulfilled. In (4.32), instead of A, we substitute $-c/b$; then we obtain

$$-4\frac{c^2+b}{b}\mu \geqslant -\frac{c}{b}\left[(c^2+b)^2+\frac{2\mu(c^2+b)}{c}+\frac{\mu^2}{c^2}\right]$$

or

$$-\frac{c}{b}\left[(c^2+b)-\frac{\mu}{c}\right]^2 \leqslant 0. \tag{4.33}$$

Since in the case considered, $c > 0$, and $b < 0$, then the last inequality can be fulfilled only for the condition

$$\mu = c(c^2+b). \tag{4.34}$$

Thus, if $c^2 + b < 0$, then for any x and x_1, satisfying the inequality $x^2 + x_1^2 > 0$, and for any $A \epsilon (0, -c/b)$, it results that $\dot{v} < 0$. If $c^2 + b > 0$, then there exists no such a μ for which $\dot{v}$ would be nonpositive for any $A \epsilon (0, -c/b)$. But if $c^2 + b = 0$, then it must follow from inequality (4.32) that $\mu = 0$, and the function v degenerates into $v = \frac{1}{2}x_1^2$.

Theorem 4.2

If the conditions of case 5 are fulfilled (i.e., if $b < 0$, $a > 0$) and if in addition $c^2 + b < 0$, then the null solution of system (2.15) is globally stable for any function f(x) satisfying the generalized Hurwitz conditions (2.24).

The proof of this theorem coincides with the proof of theorem 4.1. For this, just as in the proof of theorem 4.1, it is necessary to use function v_1 defined in equality (4.30) and theorem 1.1.

Consider now cases 1 and 4; i.e., the cases when $0 \leq b < 1$, $a > 0$. According to (4.8) in these cases we will have

$$A > 0. \tag{4.35}$$

Dividing inequality (4.25) by $A > 0$, we obtain inequality (4.32). Inequality (4.32) is fulfilled for sufficiently small A only for the condition that $\mu < 0$; on the other hand, for $\mu < 0$ this inequality cannot be fulfilled for sufficiently large A. Therefore, in the case considered, inequality (4.32) can be fulfilled neither for all $A > 0$ nor for every real μ.

Thus, from the reasoning shown, it follows that for system (2.15) in cases 1 and 4 (i.e., for $0 \leq b < 1$, $a > 0$) and in case 5 for $c^2 + b > 0$ (i.e., in the case when $b < 0$, $a > 0$ and $c^2 + b > 0$) it is impossible to find a positive function of the type "integral of the nonlinearity plus a quadratic form in the sought after variables," which would have a negative derivative on time for any f(x) satisfying the generalized Hurwitz conditions.

Section 9

Theorem 4.3

If the conditions of case 3 are fulfilled, i.e., if $a > 0$, $b < 0$, and if besides these

$$c^2 + b = 0, \tag{4.36}$$

then the null solution of system (2.15) is globally stable for any f(x) satisfying the generalized Hurwitz conditions (2.24).

Proof

As was shown above in the case considered, system (2.15), by the substitution of variables (4.12), is brought into form (4.13). For this case, in consequence of condition (4.36), the first equation of this system is not dependent on the other two. Therefore, along all solutions of system (2.15) in the case considered, there is fulfilled the equality

$$x_1 = x_{10}e^{-ct}, \tag{4.37}$$

where x_{10} is the value of x_1 for $t = 0$ on the solution considered. From formula (4.37) it follows that any solution of system (2.15), even if having one point in the plane $x_1 = 0$, remains in this plane also for all t, for which this solution is defined.

We will prove that any solution lying in plane $x_1 = 0$ goes to the origin for $t \to +\infty$. The behavior of these solutions on plane $x_1 = 0$ as follows from (4.12), (4.13) and (4.36) is described by the following system of two differential equations:

$$\frac{dy_1}{dt} = -z_1 + (1 + c^2)\,\alpha\left(\frac{cz_1 - y_1}{1 + c^2}\right), \quad \frac{dz_1}{dt} = y_1. \tag{4.38}$$

Into the functions considered, we introduce the coordinates of the phase space

$$v = \frac{1}{2}y_1^2 + \frac{1}{2}z_1^2 - c(1 + c^2)\int_0^x \alpha(x)\,dx. \tag{4.39}$$

The derivative of this function is equal to

$$\dot{v} = (1 + c^2)[x_1 - x + c\alpha(x)]\alpha(x) \tag{4.40}$$

by virtue of the differential equations of system (4.13), as is easily verified. But for those solutions of system (2.15) which lie wholly on surface $x_1 = 0$, from (4.40) we have

$$\dot{v} = (1 + c^2)[c\alpha(x) - x]\alpha(x). \tag{4.41}$$

We will prove that theorem 1.3 of N. P. Yerugin's work (ref. 3) is applicable to the solutions lying in plane $x_1 = 0$. Indeed, as mentioned above, the origin is unique and asymptotically stable in the sense of Lyapunov as an equilibrium point. Consider the straight line $\{x_1 = 0, x = 0\}$ passing through the origin. This line, following from the reasoning of section 3, contacts the field of direction of system (2.15) only at the point $x = y = z = 0$. Therefore, if the solution of system (2.15) lying on plane $x_1 = 0$ has a bounded polar angle for $t > 0$, then for sufficiently large t it does not intersect the line $\{x_1 = 0, x = 0\}$ and, consequently, plane $x = 0$. But then from theorem 3.1, this solution goes to the point (0, 0, 0) for $t \to +\infty$ and, consequently, is bounded for $t > 0$. Further, since the origin is a unique equilibrium point of system (2.15), any periodic motion lying on plane $x_1 = 0$ must enclose the origin and, accordingly must intersect line $\{x_1 = 0, x = 0\}$. However, from the generalized Hurwitz conditions (4.9), from (4.36) and from (4.41), it follows that for $x \neq 0$, $\dot{v} < 0$ on those solutions which lie on the plane $x_1 = 0$. Thus, on the plane $x_1 = 0$, we will find in the conditions of theorem 1.3 of the work of Yerugin (ref. 3), and by virtue of this theorem, that any solution lying on the plane $x_1 = 0$ goes to the origin for $t \to +\infty$.

We will also prove that all the remaining solutions of system (2.15) possess this quality. Consider an arbitrary trajectory $\phi(p, t)$ of system (2.15). If path $\phi(p, t)$ does not intersect plane $x = 0$ for $t > T$, where T is sufficiently large, then from theorem 3.1 we see that this path goes to the origin for $t \to \infty$. Suppose, therefore, that there exists an infinite sequence of intersection points of the path $\phi(p, t)$ with the plane $x = 0$.

Let $t = 0$, $t = t_1$, $t = t_2$, ... be instants of the intersection of path $\phi(p, t)$ with plane $x = 0$. For this we will suppose that $0 < t_1 < t_2 < \ldots$ and that for $t \in (t_k, t_{k+1})$, $\phi(p, t)$ lies in one of the half-spaces $x > 0$ or $x < 0$; i.e., suppose that 0, t_1, t_2, ... are a sequence of points of intersection of path $\phi(p, t)$ with plane $x = 0$.

We will prove that

$$|x(\varphi(p, t))| < \left(1 + \frac{2}{c} + \frac{1}{c^2}\right)\sqrt{2v(\varphi(p, t_k))} \tag{4.42}$$

for $t \in [t_k, t_{k+1}]$, where function v is defined by equality (4.39). For definiteness, we will suppose that for $t \in (t_k, t_{k+1})$, $\phi(p, t)$ lies in the half-space $x > 0$; i.e., suppose that $y(\phi(p, t_k)) \geq 0$, and if $z(\phi(p, t_k)) \leq 0$, then $y(\phi(p, t_k)) > 0$. Consider first that case when $z(\phi(p, t_k)) \leq 0$. As seen from lemma 3.1, path $\phi(p, t)$ for increasing time intersects surface $y - f(x) = 0$ for $t = \tau_1 \in (t_k, t_{k+1})$. On the time interval $t \in [t_k, \tau_1]$, y along path $\phi(p, t)$ decreases; consequently, $y(\phi(p, \tau_1)) < y(\phi(p, t_k))$. Therefore, from (3.29), (4.9) and (4.36) it follows that

$$x(\varphi(p, \tau_1)) < \frac{1}{c} y(\varphi(p, \tau_1)) < \frac{1}{c} y(\varphi(p, t_k)). \tag{4.43}$$

As was shown for the proof of lemma 3.1, path $\phi(p, t)$ intersects the surface $y - f(x) = 0$ only at one time on interval $t_k \leq t \leq t_{k+1}$, and consequently $x(\phi(p, t))$ has on the time interval $t_k \leq t \leq t_{k+1}$ the greatest value at $t = \tau_1$. Hence, also from (4.43), it follows that for $t \epsilon [t_k, t_{k+1}]$ there is fulfilled the inequality

$$x(\varphi(p, t)) < \frac{1}{c} y(\varphi(p, t_k)). \tag{4.44}$$

Since from the definition of v, $y(\phi(p, t_k)) \leq \sqrt{2v(\phi(p, t_k))}$, then (4.42) also follows from (4.44).

We go now to the case when $z(\phi(p, t_k)) > 0$. In this case, since it follows from the proof of lemma 3.3, path $\phi(p, t)$ for $t = \tau_1 \epsilon (t_k, t_{k+1})$ intersects plane $z - x = 0$, and thereupon for $t = \tau_2 \epsilon [\tau_1, t_{k+1})$ surface $y - f(x) = 0$ and crosses over into domain $\{y - f(x) < 0\}$ (so that τ_2 is understood to be the first instant after $t = \tau_1$ of the existence of an intersection of $\phi(p, t)$ with the surface $y - f(x) = 0$). We will prove that for $t \epsilon [t_k, \tau_1]$ on path $\phi(p, t)$ there is fulfilled the inequality

$$y \leqslant \left(2 + c + \frac{1}{c}\right) \sqrt{2v(\varphi(p, t_k))}. \tag{4.45}$$

Indeed, if for all $t \epsilon [t_k, \tau_1)$, the inequality

$$y < \left(1 + c + \frac{1}{c}\right) \sqrt{2v(\varphi(p, t_k))}, \tag{4.46}$$

is fulfilled, then (4.45) also follows. But if inequality (4.46) is violated on the interval $[t_k, \tau_1)$, then because it is fulfilled for $t = t_k$, a $t^* \epsilon (t_k, \tau_1)$ must exist such that

$$y(\varphi(p, t^*)) = \left(1 + c + \frac{1}{c}\right) \sqrt{2v(\varphi(p, t_k))}. \tag{4.47}$$

As is easily seen from the proof of lemma 3.3, $y(\phi(p, t))$ increases for $t \epsilon [t_k, \tau_1]$; therefore, for $t \epsilon (t^*, \tau_1)$ we will have

$$y(\varphi(p, t)) > \left(1 + c + \frac{1}{c}\right) \sqrt{2v(\varphi(p, t_k))}. \tag{4.48}$$

Since for $t \epsilon (t_k, t_{k+1})$, with increasing time, z decreases along path $\phi(p, t)$, there is fulfilled the relation

$$x(\varphi(p, \tau_1)) = z(\varphi(p, \tau_1)) < z(\varphi(p, t_k)) \leqslant \sqrt{2v(\varphi(p, t_k))}. \tag{4.49}$$

From equalities (3.7) and (3.29), from the GHC (4.9), from the conditions of (4.36) of the theorem proved, and from (4.48), the inequality

$$\frac{dy}{dx} < \frac{z}{\left(1+c+\frac{1}{c}\right)\sqrt{2v(\varphi(p, t_k))} - \left(c+\frac{1}{c}\right)x}.$$

is fulfilled for $t\epsilon(t^*, \tau_1)$ on path $\phi(p, t)$. Therefore, from (4.49) it also follows that

$$\frac{dy}{dx} < 1.$$

By integrating this inequality along path $\phi(p, t)$ for $t^* \leq t \leq \tau_1$ and using equality (4.47) and inequality (4.49), we obtain (4.45). Since $\phi(p, t)$ intersects plane $z - x = 0$ only at one time on the time interval $t_k \leq t \leq \tau_2$, then $y(\phi(p, t))$ on this interval has the greatest value for $t = \tau_1$, and consequently inequality (4.45) is fulfilled for $t\epsilon[t_k, \tau_2]$. Since $y = f(x) = cx + \alpha(x)$ because $t = \tau_2$ on trajectory $\phi(p, t)$, then from inequality (4.45) and condition (4.9) it follows that

$$x(\varphi(p, \tau_2)) < \left(1 + \frac{2}{c} + \frac{1}{c^2}\right)\sqrt{2v(\varphi(p, t_k))}. \tag{4.50}$$

We will show that for $t\epsilon[t_k, t_{k+1}]$ the inequality

$$x(\phi(p, t)) \leq x(\phi(p, \tau_2)). \tag{4.51}$$

is fulfilled. Indeed, if for $t\epsilon(\tau_2, t_{k+1})$ on $\phi(p, t)$ it results that $y \leq f(x)$, then inequality (4.50) follows immediately because x along path $\phi(p, t)$ decreases for $t\epsilon(\tau_2, t_{k+1})$. Now let $t = \tau_3\epsilon(\tau_2, t_{k+1})$ be the first instant after τ_2 of the existence of an intersection of path $\phi(p, t)$ with the surface $y - f(x) = 0$; then x along $\phi(p, t)$ decreases on the interval $\tau_2 \leq t \leq \tau_3$, and, therefore, $x(\tau_3) < x(\tau_2)$ on $\phi(p, t)$. Since x increases along path $\phi(p, t)$ on interval $[t_k, \tau_2]$, then a $t^*\epsilon(t_k, \tau_2)$ is found such that

$$x(\phi(p, t^*)) = x(\phi(p, \tau_3)).$$

Moreover, on interval $t_k \leq t \leq \tau_2$ we have $y \geq f(x)$ on path $\phi(p, t)$; consequently,

$$y(\phi(p, t^*)) \geq y(\phi(p, \tau_3)). \tag{4.52}$$

Further, since $z(\phi(p, t))$ decreases with passing time for $t\epsilon[t_k, t_{k+1}]$, then

$$z(\phi(p, t^*)) > z(\phi(p, \tau_3)). \tag{4.53}$$

Now let $\tau_4\epsilon(\tau_3, t_{k+1})$ be the first instant of time after τ_3 of the intersection of path $\phi(p, t)$ with the surface $y - f(x) = 0$. Use lemma 3.9 on the two segments of path $\phi(p, t)$ for $t\epsilon(t^*, \tau_2)$ and for $t\epsilon(\tau_3, \tau_4)$. From this lemma and inequalities (4.52) and (4.53), it follows that for the same x, y on $\phi(p, t)$ for $t\epsilon(t^*, \tau_2)$ is larger than for $t\epsilon(\tau_3, \tau_4)$; thus,

$$x(\phi(p, \tau_4)) < x(\phi(p, \tau_2)).$$

Repeating this reasoning a sufficient number of times, we will also prove inequality (4.51). And from inequalities (4.50) and (4.51), inequality (4.42) results.

Returning now to equality (4.40), we see that from the Hurwitz conditions (4.9) and from condition (4.36) of the theorem proved, it follows that

$$\frac{dv}{dt} < \frac{1+c^2}{c} | x_1 | | x |. \tag{4.54}$$

Proceeding also from inequality (4.42),

$$\frac{dv}{dt} < \frac{1+c^2}{c} | x_1 | \left(1 + \frac{2}{c} + \frac{1}{c^2}\right) \sqrt{2v(\varphi(p, t_k))}$$

on trajectory $\phi(p, t)$ for $t \epsilon [t_k, t_{k+1}]$. We assume

$$| x_{10} | \frac{1+c^2}{c} \left(1 + \frac{2}{c} + \frac{1}{c^2}\right) \sqrt{2} = \omega.$$

Accordingly, from the preceding inequality and from (4.37) we obtain

$$\frac{dV}{dt} < \omega \sqrt{v(\varphi(p, t_k))}\, e^{-ct} \tag{4.55}$$

on $\phi(p, t)$ for $t \epsilon [t_k, t_{k+1}]$.

Investigate now the differential equation

$$\frac{dV}{dt} = \omega e^{-ct} \sqrt{V}. \tag{4.56}$$

Let V(t) be any solution of this equation for which

$$V(0) > v(\phi(p, 0)). \tag{4.57}$$

We will then prove that for all $t \geq 0$ there is fulfilled the inequality

$$V(t) > v(\phi(p, t)). \tag{4.58}$$

Indeed, for $t = 0$, this inequality becomes (4.57). For $t \epsilon [0, t_1]$ the inequality given proceeds from (4.55) and (4.56) and from the fact that V(t) increases for increasing time t. Analogously, the truth of this inequality is proved for all $t \epsilon [t_k, t_{k+1}]$. Thus, inequality (4.58) is true for any $t \geq 0$ in all cases for which solutions V(t) and $\phi(p, t)$ of equation (4.56) and system (2.15) are defined. But equation (4.56) is easily integrated. We have:

$$2\sqrt{V(t)} = 2\sqrt{V(0)} + \omega c(1 - e^{-ct}). \tag{4.59}$$

Due to (4.59) there must exist a number K, for which

$$V(t) < K \tag{4.60}$$

for all $t \geq 0$. From (4.58) and (4.60) there follows inequality

$$v(\phi(p, t)) < K \tag{4.61}$$

for all $t \geq 0$.

From the definition of function v in equality (4.39), it follows that the sequence of points $\phi(p, t_k)$ is bounded. Thus, following from the theorem on the continuation of solutions (refs. 27, 28), the sequence t_k is unbounded (i.e., the solutions $\phi(p, t)$ of system (2.15) are continued on the entire positive semiaxis t). But if the sequence of points $\phi(p, t_k)$ is bounded, path $\phi(p, t)$ has an ω-limit point q by virtue of the Bolzano-Weirstrasse theorem. Since for $t \to \infty$, $x_1 \to 0$ along all solutions of system (2.15), as follows from equality (4.37), then the point q lies in the plane $x_1 = 0$. However, as proven earlier, $\phi(q, t)$ goes to the origin for $t \to +\infty$. As we will prove below, the null solution of system (2.15) is Lyapunov stable and, consequently, $\phi(p, t)$ also goes to the origin for $t \to +\infty$.

We will now prove that the null solution of system (2.15) is stable in the sense of Lyapunov. Take the point of the phase space with the coordinates x_0, y_0, z_0, $|x_0| < \epsilon$, $|y_0| < \epsilon$, and $|z_0| < \epsilon$. Let $T > 0$ be the first instant after $t = 0$ of the intersection of the path $\phi(p, t)$ with the plane $x = 0$; if none such exists, then $T = +\infty$. Just as above, it is easy to substantiate that

$$|x| < \left(1 + \frac{2}{c} + \frac{1}{c^2}\right)\varepsilon$$

on path $\phi(p, t)$ for $t \in (0, T)$. Yet, from the preceding reasoning, it is clear that for all $t > 0$ the inequality

$$v(\varphi(p, t)) < V(t) < \left[3\varepsilon + \frac{\omega c}{2}(1 - e^{-ct})\right]^2.$$

occurs. But then, from the definition of ω it follows that

$$v(\phi(p, t)) < \Omega^2\epsilon^2,$$

where Ω is some constant.

Proceeding from the last inequality, as in the above, the inequality

$$|x| < \sqrt{2}\left(1 + \frac{2}{c} + \frac{1}{c^2}\right)\Omega\varepsilon.$$

is fulfilled for all $t > 0$ on path $\phi(p, t)$. But from the form of function v, x follows:

$$\frac{1}{2}(z-x)^2+\frac{1}{2}y^2-\frac{1}{2}(1+c^2)x^2$$
$$+c(1+c^2)\int_0^x\left(\frac{1}{c}\,x-\alpha(x)\right)dx<\Omega^2\varepsilon^2.$$

Therefore, from $c^2x - cy + z = x_1$ we obtain

$$\frac{1}{2}x_1^2+x_1\left[cy-(1+c^2)x\right]+\frac{1}{2}(c^2+1)(y-cx)^2<\Omega^2\varepsilon^2.$$

However, since $|x_1| \leq |x_{10}|$ for all $t > 0$, we obtain

$$|y| < \Omega_1\sqrt{\epsilon}, \quad |z| < \Omega_1\sqrt{\epsilon},$$

from the last inequality; Ω_1 is some constant. The inequalities obtained thus prove that the null solution of system (2.15) is stable in the sense of Lyapunov. The theorem is proved.

Section 10

In this section we investigate cases 3 and 6 introduced in chapter II; i.e., the cases when $a < 0$, $0 < b \leq 1$. The following theorem occurs.

Theorem 4.4

If the conditions of cases 3 and 6 are fulfilled (i.e., if $a < 0$ and $0 < b \leq 1$) then the null solution of system (2.15) is globally stable for any nonlinear function f(x) satisfying the GHC (2.22) and (2.25).

Proof

For the proof of the theorem we will show that, in the cases considered, all the conditions of theorem 1.1 are fulfilled. As mentioned above, condition 1 of this theorem is fulfilled. For hyperplane L, figuring in condition 3, theorem 1.1, we choose a plane $x = 0$; then condition 3a will be fulfilled due to theorem 3.1.

Instead of function v appearing in the condition of theorem 1.1, we will take the function

$$v=\frac{1}{2}y^2+\frac{1}{2b}z^2. \tag{4.62}$$

Condition 3b of theorem 1.1 will be fulfilled in this case. Thus, we have only to prove that condition 3c of theorem 1.1 in the cases considered is also fulfilled, which we will now show.

Assume that

$$f(x) = -\frac{a}{b}x + \alpha(x), \tag{4.63}$$

then according to the GHC (2.22) and (2.25), function $\alpha(x)$ must satisfy the following conditions

$$\alpha(0) = 0, \; x\alpha(x) > 0 \text{ for } x \neq 0. \tag{4.64}$$

System (2.15) in the designation (4.63) takes the form

$$\frac{dx}{dt} = y + \frac{a}{b}x - \alpha(x), \quad \frac{dy}{dt} = z - x, \quad \frac{dz}{dt} = -b\alpha(x). \tag{4.65}$$

The following function of the coordinates of the phase space is introduced for consideration:

$$u = \frac{1}{2}y^2 + \frac{1}{2}\frac{1-b}{b}z^2 + \frac{1}{2}(z-x)^2. \tag{4.66}$$

The time derivative of this function, taken because of the differential equations of system (4.65), as is easily verified, is equal to

$$\dot{u} = -\frac{a}{b}x(z-x) - (1-b)x\alpha(x). \tag{4.67}$$

We now investigate the arbitrary trajectory $\phi(p, t)$ of system (2.15), of which initial point p lies in plane $x = 0$. For definiteness, we will say that point p lies in half space $y \geq 0$ of plane $x = 0$. In this case, if $z(p) \leq 0$, we will then say that $y(p) > 0$. Assume that there exists a $T > 0$ for which point $\phi(p, T)$ lies in plane $x = 0$. For this we will say that T is the first instant of time after $t = 0$ of the intersection of path $\phi(p, t)$ with plane $x = 0$; i.e., for $t \epsilon (0, T)$ on $\phi(p, t)$, $x > 0$ results.

Our theorem will be proved if we verify that on path $\phi(p, t)$

$$v(0) > v(T). \tag{4.68}$$

Now we will prove inequality (4.68).

Assume at first that $z(p) \leq 0$. Since we see that $x > 0$ for $t \epsilon (0, T)$ on path $\phi(p, t)$, then it follows from the third equation of system (4.65) that $z < 0$ for $t \epsilon (0, T)$. Therefore, proceeding from equation (4.67), function u on the time interval $0 < t < T$ decreases with increasing time along path $\phi(p, t)$.

Consequently, we have

$$u(p) > u(\phi(p,\ T)). \tag{4.69}$$

Since function u becomes v for $x = 0$, inequality (4.68) is proved in the case $z(p) \leq 0$.

Return now to the case when $z(p) > 0$. We begin moving along path $\phi(p, t)$ from point $\phi(p, T)$ in the direction of decreasing time. If $z(\phi(p, T)) > 0$, then for $t < T$ and sufficiently close to T, y along $\phi(p, t)$ decreases together with time. Since $y(p) \geq 0$ by assumption, then path $\phi(p, t)$ for decreasing time from $t = T$ intersects plane $z - x = 0$ for $t = \tau_1 < T$ and goes into domain $\{z - x < 0\}$. In this domain, y along $\phi(p, t)$ increases with decreasing time. For the following decrease of time, path $\phi(p, t)$ intersects either surface $y - f(x) = 0$, plane $z - x = 0$, or curve $y - f(x) = 0$, $z - x = 0$. In this case path $\phi(p, t)$ can intersect plane $z - x = 0$ at times following after $t = \tau_1$ (in the direction of decreasing time t) only for $y > 0$, as was proved in section 3.

If path $\phi(p, t)$ for $t < \tau_1$ intersects first the surface $y - f(x) = 0$, then we designate the instant of intersection of $\phi(p, t)$ with $y - f(x) = 0$ by τ_2; thus it is clear that $\tau_2 < \tau_1$. For further decreased time from τ_2, path $\phi(p, t)$ intersects plane $z - x = 0$ for $t = \tau_3 < \tau_2$. If path $\phi(p, t)$ first intersects plane $z - x = 0$ or curve $y - f(x) = 0$, $z - x = 0$, then this instant of intersection will be designated as τ_3 so that the instant of time τ_2 is not defined in this case. If $y(\tau_3) > f(x(\tau_3))$ on path $\phi(p, t)$, then for further decreased time path $\phi(p, t)$ intersects either surface $y - f(x) = 0$ for $t = \tau_4 < \tau_3$ or plane $x = 0$ for $t = \tau_4 = 0$.

When $z(\phi(p, T)) \leq 0$, the behavior of path $\phi(p, t)$ for decreased time from $t = T$ is the same as in the case $z(\phi(p, T)) > 0$, except that the instant of time $t = \tau_1$ is not defined since for $t < T$ and sufficiently close to T, path $\phi(p, t)$ is not found inside domain $\{z - x < 0\}$, as it is not difficult to see. As in the case $z(\phi(p, T)) > 0$, we will designate, by τ_2, τ_3, τ_4, the instants of intersection of path $\phi(p, t)$ with surface $y - f(x) = 0$, with plane $z - x = 0$ and again with the surface $y - f(x) = 0$ (or with plane $x = 0$ and then $\tau_4 = 0$).

The first instant after $t = 0$ of the intersection of the path $\phi(p, t)$ with plane $z - x = 0$ is designated as t_1. We will show that

$$x(\phi(p,\ t_1)) \geq x(\phi(p,\ \tau_3)) \tag{4.70}$$

and

$$z(\phi(p,\ t_1)) \geq z(\phi(p,\ \tau_3)). \tag{4.71}$$

In that case when $\tau_4 = 0$, relations (4.70) and (4.71) follow immediately because here the instants of time t_1 and τ_3 coincide. Suppose now that $t_1 < \tau_3$. Since z along $\phi(p, t)$ for $t \epsilon [0, T]$ decreases with increasing time, inequality (4.71) is fulfilled in the strict sense. Proceeding from the definitions of the instants of time t_1 and τ_3, in this case inequality (4.70) is also fulfilled and, moreover, in the strict sense.

On the interval of time $0 \leq t \leq t_1$ along $\phi(p, t)$, x varies monotonically and continuously; therefore, one and only one $\Theta_1 \epsilon [0, t_1]$ exists for which

$$x(\phi(p,\ \Theta_1)) = x(\phi(p,\ \tau_3)). \tag{4.72}$$

We will prove that

$$y(\phi(p,\ \Theta_1)) \geq y(\phi(p,\ \tau_3)). \tag{4.73}$$

When $\tau_4 = 0$, relation (4.73) follows because instants t_1, τ_3 and Θ_1 coincide. But let $t_1 < \tau_3$. Consider first the case when $y(\tau_3) \leq f(x(\tau_3))$ on $\phi(p, t)$. As was proved for lemma 3.3, the inequality $y - f(x) > 0$, from which (4.73) also proceeds, is fulfilled on path $\phi(p, t)$ for $t \epsilon (0, t_1)$. Now let $y(\tau_3) > f(x(\tau_3))$ on path $\phi(p, t)$. In this case, the instant of time τ_4 is defined. Since from the condition $t_1 < \tau_3$, it is clear that $t_1 < \tau_4$, and, consequently, $x(\tau_4) > 0$ on $\phi(p, t)$. From the definition of the instant τ_4, we have $y(\tau_4) = f(x(\tau_4))$ on path $\phi(p, t)$. Following from the same definition of the moment of time τ_4, the inequality

$$x(\tau_4) < x(\tau_3) \tag{4.74}$$

is fulfilled on path $\phi(p, t)$.

From inequalities (4.70) and (4.74), it follows that a $\Theta_2 \epsilon (0, t_1)$ exists for which

$$x(\phi(p,\ \Theta_2)) = x(\phi(p,\ \tau_4)). \tag{4.75}$$

Since $z(\phi(p, t))$ decreases with increasing time for $t \epsilon (0, T)$, it is clear that

$$z(\phi(p,\ \Theta_2)) > z(\phi(p,\ \tau_4)). \tag{4.76}$$

Moreover, $y \geq f(x)$ on path $\phi(p, t)$ on the interval of time $0 \leq t \leq t_1$; therefore, from the definition of the instants τ_4 and Θ_2, it follows that

$$y(\phi(p,\ \Theta_2)) \geq y(\phi(p,\ \tau_4)). \tag{4.77}$$

Now we apply lemma 3.9 to the intervals $\Theta_2 \leq t \leq \Theta_1$ and $\tau_4 \leq t \leq \tau_3$ on path $\phi(p, t)$, from which inequality (4.73) proceeds.

We will investigate x, z and u on path $\phi(p, t)$ as functions of the y component. To avoid the ambiguity arising for this case, the functions x, z and u will be given the following indexes: on interval $0 \leq t \leq t_1$, the index 1; on interval $\tau_3 \leq t \leq \tau_1$, the index 2 (if the instant τ_1 is not defined; i.e., if $z(\phi(p, T)) \leq 0$, then index 2 is used for the interval $\tau_3 \leq t \leq T$); and on the interval $\tau_1 \leq t \leq T$, the index 3. Since y along $\phi(p, t)$ varies monotonically on the respective intervals, the uniqueness will be completely restored in this way.

We will now prove the inequality

$$u_1(y(p)) > u_2(0). \tag{4.78}$$

Consider first the case when $y(p) \geq y(\phi(p, \tau_3))$. Because z decreases along $\phi(p, t)$ with the passing of time for $t \epsilon [0, T]$, it follows from inequality $y(p) \geq y(\phi(p, \tau_3))$ and from the form of the function u that

$$u(p) > u(\phi(p,\ \tau_3)). \tag{4.79}$$

For $t \epsilon (\tau_3, \tau_1)$ on path $\phi(p, t)$, $z - x < 0$; thus, following from (4.67), function u decreases along $\phi(p, t)$ for $t \epsilon (\tau_3, \tau_1)$. It is easily seen that $y(\phi(p, \tau_1)) < 0$; therefore, (4.78) follows from (4.79).

Assume now that $y(p) < y(\phi(p, \tau_3))$. From inequality (4.73) it follows that the inequality

$$x_1(y(\tau_3)) \leq x_2(y(\tau_3)). \tag{4.80}$$

is fulfilled on $\phi(p, t)$. We will prove that the inequality

$$x_1(y) < x_2(y) \tag{4.81}$$

is fulfilled for $y(p) < y < y(\phi(p, \tau_3))$. Since x along path $\phi(p, t)$ increases with increasing time for $y > f(x)$, inequality (4.81) can be violated only for the condition that $y \leq f(x)$. Suppose that for $y = y^*$, inequality (4.80) is violated. Since x_1 and x_2 are continuous functions of y, we can then say

$$x_1(y^*) = x_2(y^*). \tag{4.82}$$

However, we saw above that $y^* > f(x_1)$ and, on the other hand, $y^* \leq f(x_2)$. This contradicts equality (4.82): the contradiction obtained proves inequality (4.81).

Divide equality (4.67) by the third equation of system (2.15), thus obtaining

$$\frac{du}{dy} = -\frac{a}{b}x - \frac{1-b}{z-x}x\alpha(x). \tag{4.83}$$

From this equality and from inequality (4.81), it follows that

$$\frac{du_1}{dy} < \frac{du_2}{dy} \tag{4.84}$$

for $y(p) < y < y(\phi(p, \tau_3))$. Note now that from inequality (4.80), from the fact that z along $\phi(p, t)$ decreases for $t \epsilon [0, T]$ and from the form of function u, consequently

$$u_1(y(\tau_3) \geq u_2(y(\tau_3)). \tag{4.85}$$

By integrating inequality (4.84) from $y(\phi(p, \tau_3))$ to $y(p)$ and then using inequality (4.85), we obtain

$$u_1(y(p)) > u_2(y(p)). \tag{4.86}$$

When $y(p) = 0$, inequality (4.78) results. But if $y(p) > 0$, then it is clear that $u_2(y)$ decreases along with y for $0 \leq y \leq y(p)$; therefore, (4.78) follows from (4.86).

If $z(\phi(p, T)) \leq 0$ (i.e., if the instant of time τ_1 is not defined) then on the interval of time $\tau_3 \leq t \leq T$, function u along path $\phi(p, t)$ decreases with increasing time, as follows from equality (4.67). Therefore,

$$u(\phi(p, T)) < u_2(0). \tag{4.87}$$

Hence, from (4.78) it follows that

$$u(p) > u(\phi(p, T)), \tag{4.88}$$

and (4.68) proceeds from this inequality.

Suppose now that $z(\phi(p, T)) > 0$; in the case investigated, the instant τ_1 is defined. We will prove that in this case

$$u_3(y(T)) = u(\phi(p, T)) < u_2(0). \tag{4.89}$$

Indeed, x decreases for $y \leq 0$ and $x \geq 0$ along path $\phi(p, t)$; therefore, we can write

$$x_3(y) < x_2(y) \tag{4.90}$$

for all y, except $y(\phi(p, \tau_1))$, for which $x_2(y) = x_3(y)$. From inequality (4.90) and from equality (4.83) it follows that

$$\frac{du_3}{dy} < \frac{du_2}{dy} \tag{4.91}$$

for $y(\phi(p, \tau_1)) < y < y(\phi(p, T))$. Integrating inequality (4.91) from $y(\phi(p, \tau_1))$ to $y(\phi(p, T))$, we obtain

$$u_3(y(T)) = u(T) < u_2(y(T)) \tag{4.92}$$

on path $\phi(p, t)$. Since $z - x \leq 0$ for $t \epsilon [\tau_3, \tau_1]$ on $\phi(p, t)$, then function u decreases along $\phi(p, t)$ on this interval. Therefore, from (4.92) we see that

$$u(\phi(p, T)) < u_2(0). \tag{4.93}$$

Thus, (4.88) and also (4.68) follow from (4.78).

Now we will prove that the null solution of system (2.15) is stable in the sense of Lyapunov in the case investigated. We take an arbitrary $\epsilon > 0$ for which a $\delta \epsilon (0, \epsilon)$ is found such that $|f(x)| < \epsilon$ for $|x| < \delta$.

Consider the arbitrary point p with coordinates x_0, y_0, z_0, $\max \{|x_0|, |y_0|, |z_0|\} < \delta$; for definiteness assume that $x_0 > 0$. Let $T > 0$ be the first instant of time after $t = 0$ of the intersection of path $\phi(p, t)$ with plane $x = 0$. (It certainly can happen that $\phi(p, t)$ for $t > 0$ will not intersect plane $x = 0$; then we will assume $T = +\infty$. We will investigate path $\phi(p, t)$ for $t \epsilon (0, T)$. Since z decreases for $t \epsilon (0, T)$ along $\phi(p, t)$, it is clear that $z < \delta$ on the interval of path $\phi(p, t)$ considered. The y component on $\phi(p, t)$ increases for $z - x > 0$; consequently, it can increase only for $x < \delta$. Therefore, from equality $\frac{dy}{dx} = \frac{z - x}{y - f(x)}$, it follows that $dy/dx < 1$ for $y \geq 2\epsilon$. Integrating this inequality, we obtain the result that $y < 3\epsilon$ on the segment considered. But the minimum of the ordinate, as mentioned above, lies on the plane $z - x = 0$. Thus, from the decreasing of function u along $\phi(p, t)$ for $z - x < 0$ it follows that $y > -4\epsilon$ for $t\epsilon(0, T)$. From the same considerations, it is easy to conclude that $z > -4\epsilon$ for $t\epsilon(0, T)$. Since x decreases along $\phi(p, t)$ for $y < f(x)$, then $x < \max\left\{\delta, -\frac{3b\varepsilon}{a}\right\}$ follows from equality (4.64). In this way we proved that on the segment of path $\phi(p, t)$ considered, the inequality

$$\max\{|x|, |y|, |z|\} < \max\left\{-\frac{3b}{a}\varepsilon, 4\varepsilon\right\}$$

is fulfilled. Consequently, from inequality (4.68) we see that the null solution of system (2.15) is Lyapunov stable. Thus, all the conditions of theorem 1.1 in the case considered are fulfilled, which also proves the theorem.

Section 11

In section 8 we established that for system (2.15) in cases 1 and 4 for $0 \leq b < 1$, $\alpha > 0$ and in case 5 for $c^2 + b > 0$ (i.e., in the case when $b < 0$, $\alpha > 0$, $c^2 + b > 0$), it is impossible to find a positive function of the type "integral of the nonlinearity plus a quadratic form of the sought after functions" which would have a negative time derivative for any f(x) satisfying the GHC. In this section we give two conditions sufficient for the global stability of the null solution of system (2.15).

Theorem 4.5

Assume that the following conditions are fulfilled: $\alpha > 0$, $0 \leq b < 1$, or $\alpha > 0$, $b < 0$, $c^2 + b > 0$. Assume moreover that for any real $x \neq 0$ the inequality

$$0 < x\alpha(x) \leqslant \frac{1}{c}x^2 \tag{4.94}$$

is fulfilled, where c is given by formula (3.29). Then the null solution of system (2.15) is globally stable.

Proof

For the proof, we will introduce for consideration the following functions of the coordinates of the phase space:

$$v = \frac{1}{2}x_1^2 + \frac{1}{2}\frac{c^2+b}{1-b}y_1^2 + \frac{1}{2}\frac{c^2+b}{1-b}z_1^2 - c(c^2+b)\int_0^x \alpha(x)dx, \tag{4.95}$$

where the variables x_1, y_1, and z_1 are related to the variables x, y and z by formula (4.12). The time derivative of function v by virtue of the differential equations of system (2.15) is equal to

$$\dot{v} = -cx_1^2 + c(c^2 + b)\alpha^2(x) - (c^2 + b)x\alpha(x). \tag{4.96}$$

It is easy to ascertain that function v is positive definite and infinitely large (ref. 29). From inequality (4.94) it follows that the derivative $\dot{v}$ of the function v on time is nonpositive. The derivative $\dot{v}$ can become zero only on the plane $x_1 = 0$. It is immediately clear from the differential equations of system (2.15) that a positive semitrajectory of system (2.15) can never lie entirely on plane $x_1 = 0$. Therefore, we come to the conditions of theorem 4 in reference 29. Consulting this theorem thus completes the proof.

Theorem 4.6

Assume that the conditions of cases 1, 4 or 5 are fulfilled; i.e., that inequalities $a > 0$, $0 \le b < 1$, or $a > 0$, $b < 0$ are fulfilled. Assume further that the GHC (4.8) or (4.9), depending on the numbers a and b, are fulfilled. Beyond that, assume that the function $\alpha(x)$ is differentiable for all real x and there is fulfilled the inequality

$$\frac{d\alpha}{dx} > 0 \tag{4.97}$$

for all real x. Then the null solution of system (2.15) is globally stable.

Proof

Consider the following function:

$$v = \frac{1}{2}(z - x)^2 + \frac{1}{2}y^2 - (1 - b)y\alpha(x) + c(1 - b)x\alpha(x) + \frac{1}{2}(1 - b)\alpha^2(x) - c(1 - b)\int_0^x \alpha(x)\,dx. \tag{4.98}$$

Because of the differential equations of system (2.15), the time derivative of this function is equal to

$$\dot{v} = -(1 - b)\alpha'(x)(y - f(x))^2. \tag{4.99}$$

Now we will prove that the conditions of theorem 1.1 are fulfilled in the case considered. Indeed, conditions 1 and 2 of this theorem are obviously fulfilled. For the hyperplane L figuring in condition 3 of theorem 1.1, we choose plane $x = 0$; then, as follows from theorem 3.1, condition 3a will also be fulfilled. For the function v figuring in condition 3b of theorem 1.1, choose function $y^2 + z^2$; as follows from relations (4.98) and (4.99), all the rest of the conditions of theorem 1.1 will also be fulfilled. Consequently, the theorem is proved.

Section 12

In this and in the following sections of the present chapter, we will consider system (2.8) for the condition $d \neq 0$, and we will formulate some sufficient conditions for the global stability of this system's null solution. As before, we designate by A and B the real roots of the equation

$$d\omega^2 + (c - b)\omega - a = 0, \tag{4.100}$$

so that

$$A = \frac{-(c-b) + \sqrt{(c-b)^2 + 4ad}}{2d},$$
$$B = \frac{-(c-b) - \sqrt{(c-b)^2 + 4ad}}{2d}. \tag{4.101}$$

First consider cases 9, 10 and 20, for which the GHC has the single form (2.29). Instead of f(x) we introduce the new nonlinear function $\gamma(x)$ by the formula

$$\gamma(x) = f(x) - Bx. \tag{4.102}$$

Then the GHC in our cases is of the following form:

$$-\frac{a}{b} - B < \frac{\gamma(x)}{x} < 0 \quad \text{for} \quad x \neq 0. \tag{4.103}$$

System (2.8) is then rewritten in the form

$$\frac{dx}{dt} = y - Bx - \gamma(x); \; \frac{dy}{dt} = z - cx - dBx - d\gamma(x);$$
$$\frac{dz}{dt} = -ax - bBx - b\gamma(x). \tag{4.104}$$

For system (4.104) with condition (4.103) we try to find a Lyapunov function in the form "integral of the nonlinearity plus a quadratic form in the sought after functions." For this we will make use of the method presented in section 7. With system (4.104), consider the following linear system of equations with constant coefficients:

$$\frac{dx}{dt} = y - Bx - \Gamma x; \frac{dy}{dt} = z - cx - dBx - a\Gamma x;$$
$$\frac{dz}{dt} = -ax - bBx - b\Gamma x, \tag{4.105}$$

where constant Γ obeys the inequality

$$-\frac{a}{b} - B < \Gamma < 0. \tag{4.106}$$

Assume

$$k^2 = c + dB. \tag{4.107}$$

From inequality (2.43), which is true in the cases considered, it follows that $k^2 > 0$ in these cases.

Substitute in systems (4.104) and (4.105) the following change of variables:

$$x_1 = B^2x - By + z;\ y_1 = z - k^2x;\ z_1 = ky,$$
$$x = -\frac{x_1 - y_1 + \frac{B}{k}z_1}{B^2 + k^2}. \tag{4.108}$$

Then systems (4.104) and (4.105) go into the forms

$$\left.\begin{aligned}\frac{dx_1}{dt} &= -Bx_1 - (B^2 - dB + b)\gamma(x),\\ \frac{dy_1}{dt} &= -kz_1 + (k^2 - b)\gamma(x),\\ \frac{dz_1}{dt} &= ky_1 - dk\gamma(x),\end{aligned}\right\} \tag{4.109}$$

and

$$\left.\begin{aligned}\frac{dx_1}{dt} &= -Bx_1 - (B^2 - dB + b)\Gamma x,\\ \frac{dy_1}{dt} &= -kz_1 + (k^2 - b)\Gamma x,\\ \frac{dz_1}{dt} &= ky_1 - dk\Gamma x.\end{aligned}\right\} \tag{4.110}$$

For system (4.110) we will find a Lyapunov function of the form

$$v = \frac{1}{2}b_{11}x_1^2 + \frac{1}{2}b_{22}y_1^2 + \frac{1}{2}b_{33}z_1^2 + b_{12}x_1y_1 + b_{13}x_1z_1 + b_{23}y_1z_1 + \frac{1}{2}\mu\Gamma x^2, \tag{4.111}$$

where the numbers b_{ij} and μ are not defined for the present. We will require that the time derivative of this function taken by virtue of the differential equations of system (4.110) be nonpositive for any Γ, thus satisfying inequality (4.106):

$$\begin{aligned}\dot{v} = &-b_{11}Bx_1^2 - b_{11}(B^2 - dB + b)\Gamma xx_1 - b_{22}ky_1z_1\\ &+ b_{22}(k^2 - b)\Gamma xy_1 + b_{33}ky_1z_1 - b_{33}dk\Gamma xz_1 - b_{12}Bx_1y_1\\ &- b_{12}(B^3 - dB + b)\Gamma xy_1 - b_{12}kx_1z_1 + b_{12}(k^2 - b)\Gamma xx_1\\ &- b_{13}Bx_1z_1 - b_{13}(B^2 - dB + b)\Gamma xz_1 + b_{13}kx_1y_1 - b_{13}dk\Gamma xx_1\\ &- b_{23}kz_1^2 + b_{23}(k^3 - b)\Gamma xz_1 + b_{23}ky_1^2 - b_{23}dk\Gamma xy_1\\ &+ \mu\Gamma x(y - Bx - \Gamma x).\end{aligned} \tag{4.112}$$

Since $\dot{v}$ must be negative for any Γ satisfying inequality (4.106), it obviously must be negative definite for $\Gamma = 0$. Consequently, from the last equality it must be that

$$\left.\begin{array}{c} b_{23} = 0, \ b_{22} = b_{33}, \\ -b_{12}B + b_{13}k = 0, \ -b_{12}k - b_{13}B = 0. \end{array}\right\} \tag{4.113}$$

From the last equalities, it also follows that

$$b_{12} = b_{13} = 0. \tag{4.114}$$

We call $b_{22} = b_{33} = \nu$. From equalities (4.111), (4.112), (4.113) and (4.114) we obtain

$$v = \frac{1}{2} b_{11}x_1^2 + \frac{1}{2} \nu y_1^2 + \frac{1}{2} \nu z_1^2 + \frac{1}{2} \mu\Gamma x^2 \tag{4.115}$$

and

$$\begin{aligned} \dot{v} = &- b_{11}Bx_1^2 - b_{11}(B^2 - dB + \upsilon)\Gamma xx_1 \\ &+ \nu(k^2 - b)\Gamma xy_1 - \nu dk\Gamma xz_1 + \mu\Gamma x(y - Bx - \Gamma x). \end{aligned} \tag{4.116}$$

In the following, the case will be of particular interest when $k^2 \neq b$. From (4.116), it follows that in this case $b_{11} \neq 0$ since for $b_{11} = 0$ function $\dot{v}$ is of indefinite sign, as is easily seen. In addition, from inequality (2.29) it follows that $B > 0$; therefore, it must be that $b_{11} > 0$. Since function v is of interest only to within a factor of a positive constant, it can be assumed that $b_{11} = 1$. Then functions v and $\dot{v}$ are rewritten in the form

$$v = \frac{1}{2} x_1^2 + \frac{1}{2} \nu y_1^2 + \frac{1}{2} \nu z_1^2 + \frac{1}{2} \mu\Gamma x^2, \tag{4.117}$$

$$\begin{aligned} v = &- Bx_1^2 - (B^2 - dB + b)\Gamma xx_1 + \nu(k^2 - b)\Gamma xy_1 \\ &- \nu dk\Gamma xz_1 + \mu\Gamma x\left(\frac{z_1}{k} - Bx - \Gamma x\right) \end{aligned} \tag{4.118}$$

For $x_1 = 0$, the coefficient for Γx in function $\dot{v}$ must be proportional to x, since in the opposite case, for small $|\Gamma|$, $\dot{v}$ will have a sign. Therefore,

$$\frac{B}{k}\nu(b - k^2) = -\nu dk + \frac{\mu}{k}.$$

Thus, we obtain

$$\mu = \nu(bB - Bk^2 + dk^2). \tag{4.119}$$

Substituting the value of μ found into (4.118), we obtain

$$v = - Bx_1^2 - (B^2 - dB + b)\Gamma xx_1 + \nu(k^2 - b)\Gamma xy_1 +$$

$$+\frac{\nu B(b-k^2)}{k}\Gamma x z_1 - \nu B(bB - Bk^2 + dk^2)\Gamma x^2$$
$$- \nu(Bb - Bk^2 + dk^2)\Gamma^2 x^2.$$

Here, substituting $x_1 - y_1 + \frac{B}{k} z_1$ for $(B^2 + k^2)x$, we acquire

$$\dot{v} = -Bx_1^2 + [\nu(k^2 - b) - (B^2 - dB + b)]\Gamma x x_1$$
$$+ [\nu(b - k^2)(B^2 + k^2) - \nu B(bB - Bk^2 + dk^2)]\Gamma x^2$$
$$- \nu(Bb - Bk^2 + dk^2)\Gamma^2 x^3$$

or

$$\dot{v} = -Bx_1^2 + [\nu(k^2 - b) - (B^2 - dB + b)]\Gamma x x_1$$
$$+ \nu k^2(b - k^2 - dB)\Gamma x^2 - \nu(Bb - Bk^2 + dk^2)\Gamma^2 x^2. \tag{4.120}$$

The condition for nonpositivity of the last function consists of the fulfillment of the inequality

$$-4B\nu k^2(b - k^2 - dB)\Gamma + 4B\nu(Bb - Bk^2 + dk^2)\Gamma^2$$
$$\geqslant [\nu(k^2 - b) - (B^2 - dB + b)]^2\Gamma^2. \tag{4.121}$$

Utilizing this condition we will prove the following theorem.

Theorem 4.7

If the conditions of case 10 are fulfilled, i.e., if $d < 0$, $b > 0$, and $b > -\frac{\alpha}{b} > \max\{A, 0\}$, then the null solution of system (2.8) is globally stable for any nonlinear function $f(x)$ satisfying the GHC (2.29).

Proof

From the conditions of the theorem we see that $\alpha < 0$. But then

$$b - k^2 > 0. \tag{4.122}$$

Indeed, from (4.107) and (4.109) it follows that

$$Bb - Bk^2 = Bb - cB - dB^2 = -\alpha. \tag{4.123}$$

And, consequently (4.122) derives from $\alpha < 0$, and $B > 0$. Proceeding from (4.122) and condition $d < 0$, inequality (4.121) is fulfilled for any $\nu > 0$ and sufficiently small in absolute value negative Γ. Therefore, if we find a $\nu > 0$ such that inequality (4.121) is fulfilled for $\Gamma = -\frac{\alpha}{b} - B$, then this inequality will be fulfilled for such ν for all Γ satisfying inequality (4.106). Note that from (4.123) there follows

$$-\frac{a}{b}-B=-\frac{Bk^2}{b}. \tag{4.124}$$

Now, substituting $-Bk^2/b$ for Γ in inequality (4.121), after dividing by B^2k^4/b^2, we obtain

$$4b\nu(b-k^2-dB)+4B\nu(Bb-Bk^2+dk^2) \geqslant [\nu(k^2-b)-(B^2-dB+b)]^2$$

or

$$4\nu(b-k^2)(B^2-dB+b) \geq [\nu(k^2-b)-(B^2-dB+b)]^2. \tag{4.125}$$

and therefore we obtain

$$[\nu(k^2-b)+(B^2-dB+b)]^2 \leq 0. \tag{4.126}$$

The last inequality can be fulfilled if, and only if,

$$\nu=\frac{B^2-dB+b}{b-k^2}. \tag{4.127}$$

From (4.122) and from the conditions of the theorem, consequently, the ν chosen in this way is positive. Yet, as mentioned above, inequality (4.121) is fulfilled for all $\Gamma\epsilon\left(-\frac{a}{b}-B, 0\right)$ and, moreover, in the strict sense. Thus, from the proof of lemma 4.2, it follows that the derivative on time of the function

$$v_1=\frac{1}{2}x_1^2+\frac{1}{2}\frac{B^2-dB+b}{b-k^2}y_1^2+\frac{1}{2}\frac{B^2-dB+b}{b-k^2}z_1^2 + (bB-Bk^2+dk^2)\frac{B^2-dB+b}{b-k^2}\int_0^x \gamma(x)\,dx \tag{4.128}$$

taken by virtue of the differential equations of system (4.109), is nonpositive and can reduce to zero only on the straight line $x_1 = x = 0$. Therefore, in the same way used to prove theorem 4.1, we see that the null solution of system (2.8) in the case considered is globally stable for any f(x) satisfying the GHC (2.29). The theorem is proved.

Theorem 4.8

If the conditions of case 9 are fulfilled (i.e., if $d < 0$, $b > 0$, and $B > -\frac{d}{b} = A = 0$), then the null solution of system (2.8) is globally stable for any nonlinearity satisfying the GHC (2.28).

Proof

In the case considered, A. P. Tuzov (ref. 12) constructed for system (2.8) a Lyapunov function which, in our designations, has the form

$$v = \frac{1}{2} y_1^2 + \frac{1}{2} z_1^2 + dk^2 \int_0^x \gamma(x)\, dx.$$

Then its derivative on time, taken because of the differential equations of system (2.8) as is easily verified, is equal to

$$\dot{v} = -dk^2\gamma(x)(Bx + \gamma(x)).$$

In consequence of the GHC (4.103), v is positive definite, and $\dot{v}$ is negative definite; in addition, $\dot{v}$ goes to zero only for $x = 0$.

However, in the case given it is clear that all the conditions of theorem 1.1 are fulfilled if the plane $x = 0$ is taken for the hyperplane L figuring in the conditions of this theorem, and if the function $0.5y_1^2 + 0.5z_1^2$ is chosen for $x = 0$ instead of the function v. Thus the theorem is proved.

Theorem 4.9

If the conditions of case 20 are fulfilled, i.e., if $d > 0$, $b > 0$ and $B > -\frac{a}{b} > 0$, then the null solution of system (2.8) is globally stable for any nonlinear function f(x) satisfying the GHC (2.39).

Proof

In the case considered, we will prove first that there takes place the inequality

$$b - dB > 0. \tag{4.129}$$

By hypothesis, we have

$$\frac{-(c-b) - \sqrt{(c-b)^2 + 4ad}}{2d} > -\frac{a}{b} > 0$$

or

$$b\sqrt{(c-b)^2 + 4ad} < 2ad - b(c-b). \tag{4.130}$$

Squaring inequality (4.130) and collecting similar terms, we obtain

$$4a^2d^2 - 4abcd > 0.$$

However, from condition $-\frac{a}{b} > 0$ it follows that $a < 0$, and that is why $ad < bc$. But then from (4.130) we find that $b + c > 0$.

We see further that

$$b - dB = \frac{b + c + \sqrt{(c-b)^2 + 4ad}}{2},$$

and thus inequality (4.129) results.

Immediately proceeding from the relations (4.101) and (4.102),

$$b - k^2 - dB = b - c - 2dB = \sqrt{(c-b)^2} + 4ad \geq 0. \tag{4.131}$$

Assume first that

$$(c - b)^2 + 4ad > 0. \tag{4.132}$$

Then from (4.131) it results that inequality (4.121) is fulfilled for sufficiently small in absolute value negative Γ, if only $\nu > 0$. In the hypotheses of the theorem being proved, it is easy to see that relations (4.122) and (4.124) take place. But then from inequality (4.129) it follows that the right side of equality (4.127) is positive. It is known that inequality (4.121) is fulfilled for all $\Gamma \epsilon \left(-\frac{a}{b} - B,\ 0\right)$ and, moreover, in the strict sense.

Consequently, the function v defined by equality (4.128) is a Lyapunov function for system (2.8) and for the case considered. Therefore, as for the proof of theorem 4.7, it is easy to see that the null solution of system (2.8) is globally stable.

Assume now that $(c - b)^2 + 4ad = 0$. Then from equality (4.131) we obtain

$$b - k^2 - dB = 0. \tag{4.133}$$

Thus, in the case considered it follows that inequality (4.121) takes on the form

$$\begin{gathered} 4B\nu(Bb - Bk^2 + dk^2) \geqslant \nu^2(k^2 - b)^2 + (B^2 - dB + b)^2 \\ - 2\nu B(k^2B - dk^2 - bB) \end{gathered}$$

or

$$[\nu(k^2 - b) + (B^2 - dB + b)]^2 \leq 0. \tag{4.134}$$

Therefore, as above, it results that function v_1 defined by equality (4.128) is a Lyapunov function for system (2.8) and in the considered case. The derivative of this function on time due to system (2.8) as follows from inequalities (4.120) and (4.133) is equal to

$$\dot{v}_1 = -Bx_1^2 - 2(B^2 - dB + b)x_1\gamma(x) - \nu(Bb - Bk^2 + dk^2)\gamma^2(x).$$

Hence, from equalities (4.134) and (4.127), it is easy to see that

$$\dot{v}_1 = -\frac{1}{B}[Bx_1 + (B^2 - dB + b)\gamma(x)]^2. \tag{4.135}$$

In consequence of the last equality, $\dot{v}$ goes to zero only on the surface

$$Bx_1 + (B^2 - dB + b)\gamma(x) = 0. \tag{4.136}$$

Consider the path $\phi(p, t)$ of system (2.8) beginning at point $p \neq (0, 0, 0)$ which lies on the plane $x = 0$. Let $T > 0$ be the first instant after $t = 0$ of the intersection of path $\phi(p, t)$ with plane $x = 0$. We will prove that

$$v_1(\phi(p, T)) < v_1(p). \tag{4.137}$$

Inequality (4.137) cannot be fulfilled only when path $\phi(p, t)$ for all $t \epsilon (0, T)$ lies on surface (4.136). But then it is obviously necessary that

$$x_1(p) = x_1(\phi(p, T)) = 0. \tag{4.138}$$

From the first equation of system (4.109) we have in this case

$$x_1(\varphi(p, T)) = (B^2 - dB + b)\, e^{-BT} \int_0^T \gamma(x)\, e^{Bt}\, dt. \tag{4.139}$$

But on the interval of time $0 < t < T$, x on path $\phi(p, t)$ keeps the same sign by definition of the instant of time T. And, at the same time, $\gamma(x)$ on path $\phi(p, t)$ for $t \epsilon (0, T)$ keeps the same sign as a consequence of the GHC (4.103). In addition, from equality (4.139) it follows that $x_1(\phi(p, T)) \neq 0$. This proves that equality (4.137) is fulfilled.

Proceeding from this inequality, in the case given, all conditions of theorem 1.1 are fulfilled if for hyperplane L, the plane $x = 0$ is chosen and for the function v, the function v_1 defined by equality (4.128) is taken for $x = 0$. Thus, the theorem is proved.

Section 13

In this paragraph we consider system (2.8) in the conditions of cases 8, 11–14, 16, 18, 21 and 22. In all of these cases, the lower limit of the varying quantity $f(x)/x$ for $x \neq 0$ giving the GHC, is equal to A. Instead of f(x), introduce in these cases a new nonlinear function $\gamma(x)$ by the formula

$$\gamma(x) = f(x) - Ax. \tag{4.140}$$

System (2.8) then takes on the following form:

$$\frac{dx}{dt} = y - Ax - \gamma(x); \ \frac{dy}{dt} = z - cx - dAx - d\gamma(x);$$
$$\frac{dz}{dt} = -ax - bAx - b\gamma(x). \tag{4.141}$$

For system (4.141) we will, as earlier, look for a Lyapunov function in the form "Integral of the nonlinearity plus a quadratic form of the coordinates of the phase space." For this case, in addition to system (4.141), consider the linear system with constant coefficients,

$$\frac{dx}{dt} = y - Ax - \Gamma x; \ \frac{dy}{dt} = z - cx - dAx - d\Gamma x;$$
$$\frac{dz}{dt} = -ax - bAx - b\Gamma x, \tag{4.142}$$

where the constant Γ obeys the same inequality as that of the quantity $\gamma(x)/x$ in the corresponding GHC. Assume

$$k^2 = c + dA. \tag{4.143}$$

From the conditions of the cases considered and from inequalities (2.42) and (2.43), it results that $k^2 > 0$ in these cases.

In systems (4.141) and (4.142) we place the following change of variables:

$$x_1 = A^2x - Ay + z; \ y_1 = z - k^2x; \ z_1 = ky;$$
$$x = \frac{x_1 - y_1 + \frac{A}{k} z_1}{A^2 + k^2}. \tag{4.144}$$

Then systems (4.141) and (4.142) are rewritten in the forms

$$\frac{dx_1}{dt} = -Ax_1 - (A^2 - dA + b)\gamma(x); \ \frac{dy_1}{dt} = -kz_1 + (k^2 - b)\gamma(x);$$
$$\frac{dz_1}{dt} = ky_1 - dk\gamma(x) \tag{4.145}$$

and

$$\frac{dx_1}{dt} = -Ax_1 - (A^2 - dA + b)\Gamma x; \ \frac{dy_1}{dt} = -kz_1 + (k^2 - b)\Gamma x;$$
$$\frac{dz_1}{dt} = ky_1 - dk\Gamma x. \tag{4.146}$$

For system (4.146) we seek a Lyapunov function in the form (4.111). Changing B into A in the corresponding relations of the preceding paragraph, we will prove that for the condition

$$k^2 - b \neq 0 \tag{4.147}$$

the function v must have the form

$$v = \frac{1}{2}x_1^2 + \frac{1}{2}\nu y_1^2 + \frac{1}{2}\nu z_1^2 + \frac{1}{2}\nu(bA - Ak^2 + dk^2)\Gamma x^2. \tag{4.148}$$

And its time derivative taken because of the differential equations of system (4.146) is equal to

$$\dot{v} = -Ax_1^2 + [\nu(k^2 - b) - (A^2 - dA + b)]\Gamma x x_1 + \nu k^2(b - k^2 - dA)\Gamma x^2 - \nu(AB - Ak^2 + dk^2)\Gamma^2 x^2. \tag{4.149}$$

When $k^2 - b = 0$, function v can also have the form (4.148); however, it might not contain the term x_1^2. We ascertain that for such cases there can be realized the equality $k^2 - b = 0$,

$$k^2 - b = c + dA - b = \frac{c - b + \sqrt{(c-b)^2 + 4ad}}{2}. \tag{4.150}$$

Thus, it is clear that inequality (4.147) can be violated only for the condition that $a = 0$ since $d \neq 0$ in the cases just considered. But, as an immediate investigation will show, a can go to zero only in the conditions of cases 16 and 18. Thus, in the remaining cases of the section considered, function v and its derivative must have the forms (4.148) and (4.149).

The condition for the nonpositivity of function (4.149) occurs in the fulfillment of the following inequality:

$$-4A\nu k^2(b - k^2 - dA)\Gamma + 4A\nu(Ab - Ak^2 + dk^2)\Gamma^2 \geqslant [\nu(k^2 - b) - (A^2 - dA + b)]^2\Gamma^2. \tag{4.151}$$

Using this condition, we will prove several theorems on the global stability of the null solution of system (2.8).

Theorem 4.10

If the conditions of case 14 are fulfilled (i.e., if $d < 0$, $b < 0$, $0 < A < \frac{a}{b} < B$) and if in addition $A^2 - Ad + b < 0$, then the null solution of system (2.8) is globally stable for any nonlinearity f(x) satisfying the GHC (2.33).

Proof

In the case considered, the GHC for function $\gamma(x)$ is written in the form

$$0 < \frac{\gamma(x)}{x} < -\frac{a}{b} - A. \tag{4.152}$$

From equality (4.143) we obtain

$$b - k^2 - dA = b - c - dA - dA.$$

And from this and (4.101) we find

$$b - k^2 - dA = -\sqrt{(c-b)^2 + 4ad} < 0. \tag{4.153}$$

Proceeding from the last inequality, condition (4.151) is fulfilled for any sufficiently small positive Γ if $\nu > 0$. Therefore, if we choose $\nu > 0$ such that inequality (4.151) is fulfilled when $\Gamma = -\frac{a}{b} - A$, then for such ν, inequality (4.151) will be fulfilled for all $\Gamma \epsilon \left(0, -\frac{a}{b} - A\right)$ and, moreover, in the strict sense. From the definition of the numbers A and k^2, we see that

$$-\frac{a}{b} - A = -\frac{Ak^2}{b}. \tag{4.154}$$

By substituting $-Ak^2/b$ for Γ into the inequality (4.151), and then by dividing by A^2k^4/b^2, we obtain

$$\begin{gathered} 4b\nu(b - k^2 - dA) + 4A\nu(Ab - Ak^2 + dk^2) \\ \geqslant [\nu(k^2 - b) - (A^2 - dA + b)]^2 \end{gathered}$$

or

$$4\nu(b - k^2)(A^2 - dA + b) \geq [\nu(k^2 - b) - (A^2 - dA + b)]^2. \tag{4.155}$$

Therefore, we obtain

$$[\nu(k^2 - b) + (A^2 - dA + b)]^2 \leq 0. \tag{4.156}$$

The last inequality can be fulfilled if, and only if,

$$\nu = \frac{A^2 - dA + b}{b - k^2}. \tag{4.157}$$

From the hypotheses of the theorem it follows that the number ν thus chosen is positive. However, as mentioned above, for such ν, inequality (4.151) is fulfilled for all $\Gamma \epsilon \left(0, -\frac{a}{b} - A\right)$ and, moreover, in the strict sense. Consequently, from the proof of lemma 4.2, the time derivative of the function

$$\begin{aligned} v_1 = \frac{1}{2}x_1^2 + \frac{1}{2}\frac{A^2 - dA + b}{b - k^2}y_1^2 + \frac{1}{2}\frac{A^2 - dA + b}{b - k^2}z_1^2 \\ + (bA - Ak^2 + dk^2)\frac{A^2 - dA + b}{b - k^2}\int_0^x \gamma(x)\,dx \end{aligned} \tag{4.158}$$

taken because of the system of equations (4.145) with the condition (4.152) is nonpositive and can go to zero only on the line $x_1 = x = 0$. This also concludes the proof of the theorem.

Theorem 4.11

If the conditions of case 21 are fulfilled (i.e., if $d > 0$, $b = 0$, $A > 0$) and if, moreover, $A - d < 0$, then the null solution of system (2.8) is globally stable for any function $f(x)$ satisfying the GHC (2.40).

Proof

The GHC (2.40) can be written as

$$\frac{\gamma(x)}{x} > 0, \tag{4.159}$$

From the conditions of case 21, it follows that inequality (4.151) is fulfilled for sufficiently small positive Γ only if $\nu > 0$. For this inequality to be fulfilled for all positive Γ, it is necessary and sufficient that there be fulfilled the inequality

$$4Ak^2\nu(d - A) \geq [\nu k^2 + A(d - A)]^2$$

or

$$[\nu k^2 - A(d - A)]^2 \leq 0. \tag{4.160}$$

The last inequality can also be fulfilled there only in the case when

$$\nu = A\frac{d - A}{k^2}. \tag{4.161}$$

From the hypotheses of the theorem, consequently, the ν chosen in this way is positive. But then inequality (4.151) is fulfilled for all $\Gamma > 0$ and, moreover, in the strict sense. Thus, it follows that the time derivative of the function

$$v_1 = \frac{1}{2}x_1^2 + \frac{1}{2}A\frac{d-A}{k^2}y_1^2 + \frac{1}{2}A\frac{d-A}{k^2}z_1^2 + A(d - A)^2\int_0^x \gamma(x)\,dx \tag{4.162}$$

is nonpositive and can go to zero only for $x = x_1 = 0$. And this proves the theorem.

Now we will consider case 22.

Theorem 4.12

If the conditions of 22 are fulfilled i.e., if $d > 0$, $b < 0$, $0 < A < -\frac{a}{b}$ and if besides that $A^2 - Ad + b < 0$, then the null solution of system (2.8) is globally stable for any nonlinear function $f(x)$ satisfying the GHC (2.41).

The proof of this theorem is analogous to the proof of theorem 4.10.

Theorem 4.13

If the conditions of case 16 are fulfilled (i.e., if $d > 0$, $b > 0$, $A > \max\{0, -a/b\}$ and if $Ab - Ak^2 + dk^2 \geq 0$) then the null solution of system (2.8) is globally stable for any function f(x) satisfying the GHC (2.35).

Proof

The GHC in the case considered has the form (4.159).

Assume first that

$$(c - b)^2 + 4ad = 0. \tag{4.163}$$

In this case, we have $A = B$, and the proof of our theorem's assertion coincides with that part of the proof of theorem 4.9 devoted to the case when equality (4.163) is fulfilled.

Now let

$$(c - b)^2 + 4ad > 0. \tag{4.164}$$

Proceeding from inequality (4.164) and relation (4.153), inequality (4.151) is fulfilled for any sufficiently small positive Γ, if only $\nu > 0$. We will look for a positive ν such that inequality (4.151) is fulfilled also for sufficiently large Γ; then, as is easily seen, it will be fulfilled for all $\Gamma > 0$.

We will prove first that from inequality

$$A^2 - Ad + b \leq 0 \tag{4.165}$$

there follows the inequality

$$Ab - Ak^2 + dk^2 > 0. \tag{4.166}$$

Multiplying (4.165) by $k^2 > 0$, we obtain

$$-A^2k^2 + dAk^2 \geq bk^2.$$

Adding A^2b to both sides of inequality (4.166), we obtain

$$A^2b - A^2k^2 + dAk^2 \geq (A^2 + k^2)b > 0.$$

And this proves (4.166).

Suppose that inequality (4.165) is fulfilled in the strict sense. Further, let there be fulfilled the inequality

$$k^2 - b < 0. \tag{4.167}$$

Assume in this case that

$$\nu = \frac{A^2 - Ad + b}{k^2 - b} > 0, \tag{4.168}$$

then inequality (4.151) will be fulfilled for all $\Gamma > 0$ and, moreover, in the strict sense.

Let there be true the equality

$$k^2 - b = 0. \tag{4.169}$$

In this case, inequality (4.151) is fulfilled for all $\Gamma > 0$, if only ν is positive and sufficiently great.

Suppose now that

$$k^2 - b > 0. \tag{4.170}$$

We will prove that the following inequality is true:

$$A^2b - A^2k^2 + Adk^2 > (b - k^2)(A^2 - dA + b). \tag{4.171}$$

Indeed, if the right side of inequality (4.171) is expanded and the common terms cancelled, we obtain the inequality

$$b(b - Ad - k^2) < 0,$$

which proceeds from relation (4.153) and condition (4.164). Following from inequality (4.171), if we assume that

$$\nu = \frac{A^2 - dA + b}{b - k^2} > 0, \tag{4.172}$$

then inequality (4.151) will be fulfilled for all $\Gamma > 0$ in the strict sense.

Suppose now that

$$A^2 - Ad + b = 0. \tag{4.173}$$

In this case, inequality (4.151) is also fulfilled in the strict sense for $\Gamma > 0$, if only ν is positive and sufficiently small.

Go now to the case when

$$A^2 - dA + b > 0. \tag{4.174}$$

If inequality (4.170) is fulfilled, it is then necessary to choose ν by formula (4.168), and inequality (4.151) will be strictly fulfilled for all $\Gamma > 0$. If inequality (4.167) is fulfilled, then from inequality (4.171) it follows that inequality (4.151) also will be fulfilled in the strict sense for $\Gamma > 0$ if ν is chosen by formula (4.172). But if, finally, equality (4.169) is true, then as before, inequality (4.151) is fulfilled in the strict sense for all $\Gamma > 0$, only if ν is sufficiently great.

Thus, if (4.164) is fulfilled, a ν can always be found such that inequality (4.151) is fulfilled in the strict sense for all $\Gamma > 0$. But then, as follows from the preceding reasoning, the null solution of system (2.8) is globally stable for any f(x) satisfying the GHC (2.35). Thus the theorem is proved.

Theorem 4.14

If the conditions of case 18 are fulfilled (i.e., if $d > 0$, $b > 0$, $A = -\frac{a}{b} = 0$) then the null solution of system (2.8) is globally stable for any function f(x) satisfying the GHC (2.37).

Proof

In the case $c \neq b$, the theorem was proved by A. P. Tuzov (ref. 12). Therefore, suppose that $c = b$; for this case, in reference 12 a Lyapunov function is constructed which in our notations has the form

$$v = \frac{1}{2}(z - k^2x)^2 + \frac{1}{2}k^2y^2 + cd\int_0^x f(x)\,dx. \tag{*}$$

The time derivative of this function taken by virtue of the differential equations of system (2.8) is equal to

$$\dot{v} = -bdf^2(x).$$

Consequently, if plane $x = 0$ is taken for hyperplane L and the function defined by equality (*) is taken for the function v, then all the conditions of theorem 1.1 will be fulfilled. Thus the theorem is proved.

Consider now system (2.8) for the condition that $A^2 - Ad + b = 0$. Then the following theorem is true.

Theorem 4.15

Let the conditions of either case 14, 21, or 22 be fulfilled. Moreover, let

$$A^2 - Ad + b = 0. \tag{4.175}$$

Then the null solution of system (2.8) is globally stable for any function f(x) satisfying the GHC.

Proof

From equality (4.175) it follows that the first equation of system (4.145) is not dependent on the other two, which is why along all solutions of system (2.8) there is fulfilled the equality

$$x_1 = x_{10} e^{-At}, \tag{4.176}$$

where x_{10} is the value of x_1 for $t = 0$ on the solution considered.

We introduce for consideration the following function of the coordinates of the phase space:

$$v = \frac{1}{2} y_1^2 + \frac{1}{2} z_1^2 + (Ab - Ak^2 + dk^2) \int_0^x \gamma(x)\, dx. \tag{4.177}$$

The time derivative of this function taken because of the differential equations of system (2.8), as is easily verified, is equal to

$$\begin{aligned} \dot{v} = [&(k^2 - b) x_1 + k^2 (b - k^2 - dA) x \\ &- (Ab - Ak^2 + dk^2) \gamma(x)] \gamma(x). \end{aligned} \tag{4.178}$$

We will prove that for $x_1 = 0$, $\dot{v} \leq 0$ and $\dot{v}$ can go to zero only for $x = 0$. We have

$$\dot{v}\,|_{x_1=0} = [k^2 (b - k^2 - dA) x - (Ab - Ak^2 + dk^2) \gamma(x)] \gamma(x). \tag{4.179}$$

If the conditions of case 21 are fulfilled, then the proof of the assertion proceeds from (4.179) and from the GHC (4.159).

Now let the conditions of either case 14 or 22 be fulfilled. In both of these cases, the GHC has the form (4.152). In (4.179), substitute $\left(-\frac{a}{b} - A\right)x$ instead of $\gamma(x)$. By virtue of (4.154) we have

$$\dot{v} = \left[k^2 (b - k^2 - dA) + (Ab - Ak^2 + dk^2) \frac{Ak^2}{b}\right] \left(-\frac{Ak^2}{b}\right) x^2.$$

Consequently, from here and (4.175), $\dot{v} \equiv 0$ for $x_1 = 0$ and $\gamma(x) = \left(-\frac{a}{b} - A\right)x$. From (4.152) it then follows that $\dot{v} \leq 0$ and $\dot{v} = 0$ only for $x = 0$ if $x_1 = 0$.

The subsequent proof of theorem 4.15 is carried out just as in the proof of theorem 4.3, except that the function defined by means of equality (4.177) should be considered instead of the function v given by equality (4.39).

In this section we will formulate several conditions for the global stability of the null solution of system (2.8). The conditions these impose on f(x) are more severely restrictive than the GHC.

Theorem 4.16

Suppose that the conditions of cases 8, 11, 12 or 13 are fulfilled; suppose, moreover, that function f(x) for all $x \neq 0$ satisfies the inequality

$$A < \frac{f(x)}{x} \leqslant \frac{(A^2 + k^2)(b - k^2)}{Ab - Ak^2 + dk^2}, \tag{4.180}$$

then the null solution of system (2.8) is globally stable.

Proof

We will prove first that in the conditions of the cases considered there is true the inequality

$$A^2 - Ad + b > 0. \tag{4.181}$$

If the conditions of cases 8 or 11 are fulfilled, then inequality (4.181) is obvious. Let the conditions of cases 12 or 13 be fulfilled; i.e., let $d < 0$, $b < 0$, $0 < A < B \leq -a/b$. Then it is clear that $a > 0$ and $c - b > 0$. From the definition of the number B we have

$$\frac{-(c-b) - \sqrt{(c-b)^2 + 4ad}}{2d} \leqslant -\frac{a}{b}$$

or

$$-2ad + b(c-b) \geq -b\sqrt{(c-b)^2 + 4ad}. \tag{4.182}$$

Thus, after squaring and eliminating, we have

$$ad \leq bc. \tag{4.183}$$

From (4.182) we conclude that $-2ad + b(c-b) > 0$. Moreover, from the conditions of the cases, it follows that $(c-b)^2 + 4ad > 0$. From the last two inequalities we have $c^2 - b^2 > 0$. From here and from $c - b > 0$, we conclude

$$c + b > 0. \tag{4.184}$$

By the definition of A we have:

$$-2Ad = c - b - \sqrt{(c-b)^2 + 4ad}.$$

From here and (4.183) and (4.184) we obtain

$$-2Ad \geq (c-b) - \sqrt{(c-b)^2 + 4bc} = c - b - c - b = -2b$$

or

$$-Ad + b \geq 0,$$

This last inequality thus proves (4.181). Moreover, relation (4.153) leads to $k^2 - b > 0$ in the cases considered.

From inequality (4.180) it follows that

$$0 < \frac{\gamma(x)}{x} \leqslant \frac{k^2(b - k^2 - aA)}{Ab - Ak^2 + dk^2}. \tag{4.185}$$

But then, proceeding from relation (4.149), the time derivative of the function

$$v = \frac{1}{2}x_1^2 + \frac{1}{2}\frac{A^2 - dA + b}{k^2 - b}y_1^2$$
$$+ \frac{1}{2}\frac{A^2 - dA + b}{k^2 - b}z_1^2 + (bA - Ak^2 + dk^2)\frac{A^2 - dA + b}{k^2 - b}\int\limits_0^x \gamma(x)\,dx, \tag{4.186}$$

due to the differential equations of system (2.8), is nonpositive and can go to zero only for $x_1 = 0$. Thus the assertion of the theorem also follows.

Theorem 4.17

Let the conditions of cases 14, 21, or 22 be fulfilled. Moreover, let inequality (4.181) and condition (4.180) be fulfilled. Then the null solution of system (2.8) is globally stable.

Theorem 4.18

Let the conditions of case 16 be fulfilled. In addition, let the inequality $Ab - Ak^2 + dk^2 < 0$ be true. Then the null solution of system (2.8) is globally stable if condition (4.180) is true.

The proofs of the last two theorems coincide with the proof of theorem 4.16.

Section 15

In this section we consider case 17 and find the conditions sufficient for the global stability of the null solution of system (2.8). As will be proved in the end of chapter VII, we find that the conditions for this are necessary.

From the conditions of case 17 and from the définition of the quantity A as a root of equation (4.100), it results in this case that

$$c + dA = a + bA = 0.$$

Assume, as before, that $\gamma(x) = f(x) - Ax$; then system (2.8) assumes the following form:

$$\frac{dx}{dt} = y - Ax - \gamma(x); \quad \frac{dy}{dt} = z - d\gamma(x); \quad \frac{dz}{dt} = -b\gamma(x). \tag{4.187}$$

The GHC (2.36) in this case is written in the form $x\gamma(x) > 0$ for $x \neq 0$.

Theorem 4.19

If the conditions of case 17 are fulfilled and if function $\gamma(x)$ satisfies the conditions

$$\overline{\lim_{x \to +\infty}} \left(\gamma(x) + \int_0^x \gamma(x)\, dx \right) = +\infty \tag{4.188}$$

and

$$\overline{\lim_{x \to -\infty}} \left(-\gamma(x) + \int_0^x \gamma(x)\, dx \right) = +\infty, \tag{4.189}$$

then the null solution of system (2.8) is globally stable.

The proof of this theorem rests essentially on the following lemma.

Lemma 4.3

If the conditions of theorem 4.19 are fulfilled, then any path $\phi(p, t)$ of system (2.8) for $t \geq 0$, lying in one of the half-spaces $\{x < 0\}$ or $\{x > 0\}$, goes to the origin as $t \to +\infty$.

Proof

In the case considered, A. P. Tuzov (ref. 12) constructed a Lyapunov function which in our notations has the form

$$v=\frac{1}{2}x_1^2+\frac{1}{2}\frac{\mu}{Ab}z^2+\mu\int_0^x\gamma(x)\,dx, \tag{4.190}$$

where $x_1 = A^2x - Ay + z$ and the quantity μ is defined in the following way: $\mu = A|A^2 - dA + b|$, if $A^2 - dA + b \neq 0$, and μ is an arbitrary number in the interval $0 < \mu < 4A^3$, if $A^2 - dA + b = 0$.

The time derivative of function v taken because of the differential equations of system (4.187) is equal to

$$\dot{v}=-Ax_1^2-\left[(A^2-dA+b)+\frac{\mu}{A}\right]x_1\gamma(x)-\mu\gamma^2(x). \tag{4.191}$$

In the case considered it is not difficult to see that the function v is positive definite and its derivative is negative definite.

For definiteness, suppose that $\phi(p, t)$ lies in the half-space $x > 0$ for $t \geq 0$. According to lemma 3.1, it is true that $z > 0$ on path $\phi(p, t)$ for $t \geq 0$ (since, in the contrary case, path $\phi(p, t)$ would intersect with plane $x = 0$ for positive values of time).

Suppose first that $\int_0^{+\infty}\gamma(x)dx$ diverges. From relation (4.191) it is easy to conclude that the time derivative $\dot{v}$ of the function v is a negative definite quadratic form of the quantities x_1 and $\gamma(x)$. Since it results that $x > 0$ and $\gamma(x) > 0$ on path $\phi(p, t)$ for all $t \geq 0$, then $\dot{v} < 0$ on $\phi(p, t)$ for $t \geq 0$. Therefore, on path $\phi(p, t)$ for $t > 0$ there is fulfilled the inequality

$$v < v(p), \tag{4.192}$$

where v(p) is the value of function v at point p. Since, by supposition, the integral $\int_0^{+\infty}\gamma(x)dx$ diverges, inequality (4.192) is fulfilled only in bounded portions of the half-space $x > 0$. Therefore, path $\phi(p, t)$ is stable in the sense of LaGrange and, consequently, has an ω-limit point. Let q be the ω-limit point of trajectory $\phi(p, t)$, and we will prove that q coincides with the origin. Indeed, suppose to the contrary that this is not so. Note that because of the monotonicity of the variation of function v along path $\phi(p, t)$, there is fulfilled on this path the relation

$$\lim_{x\to+\infty} v = v(q). \tag{4.193}$$

We will pass through point q on path $\phi(q, t)$ of system (2.8). Since, by supposition, q is not the origin, a $t_1 > 0$ can be found such that

$$v(\phi(q, t_1)) < v(q). \tag{4.194}$$

However, trajectory $\phi(q, t)$ is limiting for trajectory $\phi(p, t)$, while function v is not continuous. Therefore a $t_2 > 0$ can be found such that

$$v(\phi(p, t_2)) < v(q). \tag{4.195}$$

The last inequality contradicts the fact that v decreases with increasing time along path $\phi(p, t)$, and it also contradicts relation (4.193). The contradictions obtained thus prove that point q coincides with the origin.

Since point q is any ω-limit point of path $\phi(p, t)$, this also shows that path $\phi(p, t)$ goes to the origin in the case considered.

Returning to the case when the integral $\int_0^{+\infty} \gamma(x)dx$ diverges, on the basis of the conditions of theorem 4.19, we will have

$$\overline{\lim_{x \to +\infty}}\, \gamma(x) = +\infty. \tag{4.196}$$

Return now to the function x_1. The time derivative of this function, taken because of the differential equations of system (4.187), obviously is equal to

$$\dot{x}_1 = -Ax_1 - (A^2 - Ad + b)\gamma(x).$$

Assume that

$$G = A^2 - Ad + b, \tag{4.197}$$

then

$$\dot{x}_1 = -Ax_1 - G\gamma(x). \tag{4.198}$$

Proceeding from lemma 3.3, path $\phi(p, t)$ for $t \geq 0$ goes to the origin if it lies in domain $\{x > 0,\ y - Ax - \gamma(x) \leq 0,\ z > 0\}$. Suppose, therefore, that point p lies in domain $\{x > 0,\ y - Ax - \gamma(x) > 0,\ z \geq 0\}$. We will prove that path $\phi(p, t)$, in this case for $t > 0$, intersects the surface $y - f(x) = 0$.

Two cases are possible.

1. $G \leq 0$. In this case, resulting from equality (4.198), the function x_1 on path $\phi(p, t)$ is bounded from below since it increases for negative x_1. Suppose that path $\phi(p, t)$ does not intersect surface $y = Ax + \gamma(x)$. Then for all $t \geq 0$ on $\phi(p, t)$ there is fulfilled the inequality

$$y - Ax > \gamma(x). \tag{4.199}$$

However, on path $\phi(p, t)$, z is bounded for $t > 0$ (since z decreases and is positive). We will prove that x on path $\phi(p, t)$ is not bounded. Indeed, assert to the contrary that x on $\phi(p, t)$ is bounded. But then, as proved by the equality

$$\frac{dy}{dx} = \frac{z - d\gamma(x)}{y - Ax - \gamma(x)}, \tag{4.200}$$

y on $\phi(p, t)$ is also bounded for all $t \geq 0$. Consequently, $\phi(p, t)$ is stable according to LaGrange in a positive direction and thus has an ω-limit point q with the coordinates x_0, y_0, z_0. Since x along path $\phi(p, t)$ increases and z decreases with increasing time, then along $\phi(p, t)$ there is fulfilled the relations

$$\lim_{t \to +\infty} z = z_0 \text{ and } \lim_{t \to +\infty} x = x_0 > 0. \tag{4.201}$$

We will pass through point q of path $\phi(q, t)$ of system (2.8). Since $x_0 > 0$, then for $t > 0$ and sufficiently small on $\phi(q, t)$ it results that $z < z_0$, which contradicts the fact that $\phi(q, t)$ is a limit trajectory for $\phi(p, t)$, and thus contradicts relation (4.201). Therefore, x on path $\phi(p, t)$ increases monotonically and without bound with increasing time. But, from (4.196) and (4.199), it then follows that on path $\phi(p, t)$ there is fulfilled

$$\overline{\lim_{x \to +\infty}} (y - Ax) = +\infty. \tag{4.202}$$

Since z on path $\phi(p, t)$ is bounded, then from the last relation and from the definition of x_1 it follows that on $\phi(p, t)$ there is fulfilled

$$\overline{\lim_{x \to +\infty}} x_1 = -\infty. \tag{4.203}$$

The last relation contradicts the fact that x_1 on path $\phi(p, t)$ is bounded from below for $t \geq 0$. The contradiction obtained proves that $\phi(p, t)$ for $t > 0$ intersects surface $y - Ax - \gamma(x) = 0$.

2. $G > 0$. Suppose in this case that path $\phi(p, t)$ for $t \geq 0$ remains in domain $\{x > 0,\ y - Ax - \gamma(x) > 0,\ z > 0\}$. As in the preceding case, we will prove that x on path $\phi(p, t)$ increases without bound with increasing time and that relations (4.199) and (4.202) are also fulfilled in this case. Relation (4.203), as in the preceding case, follows from the boundedness of z on path $\phi(p, t)$ and from relation (4.202).

Now let $T > 0$ be the instant of time to which the solution $\phi(p, t)$ of system (2.8) is continued. Certainly it can happen that $T = +\infty$. For the following reasoning, it does not matter whether T is a finite or infinite number. Let E be the set of those values of time t, in the half-interval $[0, T)$, when on path $\phi(p, t)$ the following inequalities are fulfilled:

$$x_1 \leqslant 0, \quad \frac{dx_1}{dt} = -Ax_1 - G\gamma(x) < 0. \tag{4.204}$$

From relation (4.203), it follows that x_1 on path $\phi(p, t)$ is not bounded below for $t \geq 0$; therefore, on path $\phi(p, t)$ it must be true that

$$\int_E \frac{dx_1}{dt} dt = \int_E dx_1 = -\infty. \tag{4.205}$$

Dividing the third equation of the system by equality (4.198), we obtain

$$\frac{dz}{dx_1} = \frac{-b\gamma(x)}{-Ax_1 - G\gamma(x)}. \tag{4.206}$$

On set E along path $\phi(p, t)$ there will be fulfilled the inequality

$$\frac{dz}{dx_1} \geqslant \frac{b}{G}. \tag{4.207}$$

Since dx_1/dt is negative on set E, from the last inequality we obtain

$$\frac{dz}{dt} \leqslant \frac{b}{G} \cdot \frac{dx_1}{dt}. \tag{4.208}$$

This inequality is true on the set E; integrating it on this set, we obtain

$$\int_E dz \leqslant \frac{b}{G} \int_E dx_1.$$

And thus, from here we obtain

$$\int_E dz = -\infty. \tag{4.209}$$

Since dz/dt on path $\phi(p, t)$ is negative for $t \geq 0$, from the last inequality we obtain

$$\int_0^T dz = -\infty. \tag{4.210}$$

The last relation contradicts the fact that z is bounded on path $\phi(p, t)$ for $t \geq 0$. The contradiction obtained thus proves that path $\phi(p, t)$ intersects the surface $y - Ax - \gamma(x) = 0$ for $t = t_1 > 0$.

We will prove now that path $\phi(p, t)$ goes to the origin. Let point p have coordinates x_0, y_0, z_0. Assume that

$$m = \max_{0 \leqslant x \leqslant x_0} |\gamma(x) + Ax|. \tag{4.211}$$

We introduce for consideration point q with coordinate $x = x_0$:

$$y = y_0 + 2a + m, \ z = z_0,$$

where

$$a = \max \{3mb, 3z_0^2, 3x_0^2, 1\}. \tag{4.212}$$

We will prove that path $\phi(q, t)$ for $t \leq 0$ intersects plane $x = 0$. For this we will prove that on the interval $0 \leq x \leq x_0$ on $\phi(q, t)$ there is fulfilled the inequality

$$y > a + m. \tag{4.213}$$

Suppose to the contrary that this is not so; i.e., suppose that there exists a $\tau < 0$ such that on path $\phi(q, t)$ it results that

$$0 \leq x(\tau) < x_0, \; y(\tau) = \alpha + m \tag{4.214}$$

and

$$x(\tau) < x \leq x_0, \; y(\tau) < y(t) \text{ for } t \epsilon (\tau, 0]. \tag{4.215}$$

Following from equality $\frac{dz}{dx} = \frac{-b\gamma(x)}{y - Ax - \gamma(x)}$, in this case for $t \epsilon [\tau, 0]$ on path $\phi(q, t)$ there is fulfilled

$$\frac{dz}{dx} > \frac{-bm}{y - m} \geqslant \frac{-bm}{\alpha}.$$

Integrating this inequality, we obtain

$$z(\tau) < z_0 + \frac{bmx_0}{\alpha}. \tag{4.216}$$

Since z decreases with increasing time on the interval $\tau \leq t \leq 0$ along $\phi(q, t)$, then from (4.200) because of (4.216) we see that

$$\frac{dy}{dx} < \frac{z_0 + \frac{bmx_0}{\alpha}}{\alpha} \tag{4.217}$$

on path $\phi(q, t)$ for $t \epsilon [\tau, 0]$.

Integrating the last inequality we obtain

$$y(\tau) > y_0 + 2\alpha + m - \frac{z_0 x_0}{\alpha} - \frac{bmx_0^2}{\alpha^2}.$$

Thus, from (4.212) we have

$$y(\tau) > y_0 + \alpha + m > \alpha + m,$$

which contradicts equality (4.214).

Therefore, path $\phi(q, t)$ intersects plane $x = 0$ for $t = t_q < 0$, and for $t \epsilon [t_q, 0]$ on it, there results

$$y > Ax + \gamma(x). \tag{4.218}$$

In exactly the same way as for path $\phi(p, t)$, we will prove that path $\phi(q, t)$ intersects the surface

$$y - Ax - \gamma(x) = 0 \text{ for } t = T_q > 0.$$

Let ζ_q be the abscissa of point $\phi(q, T_q)$. From lemma 3.9 it follows that abscissa ζ_1 is a point on $\phi(p, t_1)$ (the point of intersection between the path $\phi(p, t)$ and the surface $y - Ax - \gamma(x) = 0$) smaller than ζ_q. If path $\phi(p, t)$ for $t > t_1$ remains in domain $\{x > 0, y - Ax - \gamma(x) \leq 0, z > 0\}$, then from lemma 3.3, it goes to the origin. Suppose that path $\phi(p, t)$ for $t = t_2 > t_1$ intersects surface $y - Ax - \gamma(x) = 0$ and goes into domain $\{x > 0, y - Ax - \gamma(x) > 0, z > 0\}$. Let ζ_2 be the abscissa of point $\phi(p, t_2)$. However, z montonically decreases along path $\phi(p, t)$; therefore, $z(\phi(p, t_2)) < z_0$. Consequently, if $\zeta_2 \leq x_0$ then, by applying lemma 3.9 to paths $\phi(p, t)$ and $\phi(q, t)$, we will prove that path $\phi(p, t)$ intersects surface $y - Ax - \gamma(x) = 0$ for $t = t_3 > t_2$ and for this it is true that

$$\zeta_3 = x(\phi(p, t_3)) < \zeta_q. \tag{4.219}$$

But if $\zeta_2 > x_0$, then, since z decreases along path $\phi(p, t)$, we can apply the same lemma to the two segments of path $\phi(p, t)$ (i.e., to segment $t \epsilon [0, t_1]$ and to segment $t \geq t_2$), and prove that path $\phi(p, t)$ intersects surface $y - Ax - \gamma(x) = 0$ for $t = t_3 > t_2$ and that inequality (4.219) is fulfilled.

From the preceding it is clear that there is only one of two possibilities: either path $\phi(p, t)$ for sufficiently large t lies in domain $\{x > 0, y - Ax - \gamma(x) \leq 0, z > 0\}$ and then, according to lemma 3.3, it goes to the origin; or there exists a sequence $t_1, t_2, t_3 \ldots \to +\infty$ of the instants of intersection of path $\phi(p, t)$ with surface $y - Ax - \gamma(x) = 0$. For this we see that

$$x(\phi(p, t_k)) = \zeta_k < \zeta_q. \tag{4.220}$$

From inequality (4.220) it follows that the abscissa of point $\phi(p, t_k)$ is bounded. Since this point lies on the surface $y - Ax - \gamma(x) = 0$, from the boundedness of the abscissa and from the continuity of $\gamma(x)$, it follows that the y component of point $\phi(p, t_k)$ is also bounded. As mentioned above, z is bounded for $t \geq 0$ on path $\phi(p, t)$. Therefore, the sequence of points $\phi(p, t_k)$ is bounded and, in addition, has a limit point r.

We will prove that r coincides with point $x = y = z = 0$. Suppose to the contrary that this is not so. We pass through point r on path $\phi(r, t)$ of system (2.8). This path will be a limit for $\phi(p, t)$ and, consequently, will lie wholly in domain $\{x \geq 0, z \geq 0\}$. But if point r does not coincide with the origin, then it is easy to see that for sufficiently small $t > 0$, z will decrease with increasing time on $\phi(r, t)$. However, z decreases monotonically along path $\phi(p, t)$; therefore,

$$\lim_{t \to +\infty} z(\varphi(p, t)) = z(r). \tag{4.221}$$

This last relation contradicts the fact that $z(\phi(r, t))$ decreases with increasing time and that $\phi(r, t)$ is an ω-limit trajectory for $\phi(p, t)$. The contradiction obtained proves that point r coincides with the origin. As is easily seen, the origin establishes itself as a Lyapunov stable equilibrium position of system (2.8). Therefore, path $\phi(p, t)$ has a Lyapunov stable equilibrium position for its ω-limit points and, moreover, goes to this equilibrium position as $t \to +\infty$. Thus the lemma is proved.

For the proof of theorem 4.19, we note that the conditions of this theorem and also all the conditions of theorem 1.1 are fulfilled. Indeed, conditions 1 and 2 of this theorem are obviously fulfilled. For the hyperplane L figuring in condition 3 of the theorem, we select the plane $x = 0$; then, as follows from lemma 4.3, condition 3a will be fulfilled. For the function v figuring in condition 3b of the theorem, we select function v introduced by equality (4.190). On plane $x = 0$, this function becomes a positive definite quadratic form of coordinates y and z, and thus condition 3b of the theorem is also fulfilled. The fulfillment of condition 3c follows from equality (4.191) and from the preceding reasoning.

Therefore, all the conditions of theorem 1.1 are fulfilled, and theorem 4.19 is also proved.

Chapter V. ON THE BOUNDEDNESS OF SOLUTIONS

In this chapter we consider cases 1, 4 and 5; i.e., those cases in which there is no success in establishing the global stability of the solutions of system (2.15) for any nonlinear f(x) satisfying the GHC. Throughout the complete chapter we will suppose that function f(x) is continuously differentiable for all real x and that there exist numbers $\epsilon > 0$ and $x_0 > 0$ such that

$$\alpha'(x) = f'(x) - c > \epsilon \text{ for } |x| \geq x_0. \tag{5.1}$$

For this assumption we will prove several theorems relating to the behavior of the solutions of system (2.15).

Section 16

Introducing the following notation, we assume that

$$D = -\min \alpha'(x) \text{ for } |x| \leq x_0. \tag{5.2}$$

In the following we will suppose that $D \geq 0$, or in the opposite case the conditions of theorem 4.6 would prevail.

Designate further that

$$x_1 = \frac{10Dx_0}{\varepsilon} + 2x_0, \tag{5.3}$$

$$x_2 = 10x_1, \tag{5.4}$$

$$m = \max |f(x) + x| \text{ for } |x| \leq x_2, \tag{5.5}$$

$$R = \frac{200m^2}{x_0}. \tag{5.6}$$

We will prove the following lemma.

Lemma 5.1

Suppose that inequalities $a > 0$ and $b < 1$ are fulfilled. Suppose further that condition (5.1) is fulfilled. Let point p lie in domain $\{0 \leq x \leq x_0, z \geq -x_0, y \geq f(x)\}$; moreover, let

$$v(p) \geq \frac{1}{2} R^2, \tag{5.7}$$

where v is the function of the coordinates of the phase space introduced in equality (4.98).

Then path $\phi(p, t)$ of system (2.15) intersects plane $x = x_1$ for $t = t_1 > 0$ (by t_1 is meant the first instant of time after $t = 0$ of the intersection of $\phi(p, t)$ with plane $x = x_1$). And on path $\phi(p, t)$ are fulfilled the relations

$$y > f(x) \text{ for } t \in [0, t_1], \tag{5.8}$$

$$v(\phi(p, 0)) = v(p) > v(\phi(p, t_1)). \tag{5.9}$$

Proof

For the proof of this lemma we will consider only path $\phi(p, t)$ of system (2.15). In connection with this, we will sometimes consider different functions of the coordinates of the phase space simply as functions of time. As an example, $v(t)$ is the value of function v at point $\phi(p, t)$. For the proof of this lemma, we will look at two different cases:

$$\text{I. } \frac{y(p)}{|z(p)|} \geqslant 1, \quad \text{II. } \frac{y(p)}{z(p)} < 1.$$

First we will consider case I. Because of inequality (5.7), the form of function v in equality (4.98), the GHC (4.8) and (4.9), and designation (3.29), it is easy to verify the inequality

$$y(p) > 100\ m. \tag{5.10}$$

From inequality (5.10) it follows that for sufficiently small $t > 0$, there is fulfilled the inequality

$$y(t) > f(x(t)). \tag{5.11}$$

From the first equation of system (2.15) we see that x increases for such t along $\phi(p, t)$. We will now prove that, until $x \leq x_1$ on $\phi(p, t)$, on this path there is fulfilled the inequality

$$y > 0.9y(p). \tag{5.12}$$

From inequality (5.10) and designation (5.5), it follows that inequality (5.12) also involves inequality (5.11). Accordingly, if we establish the truth of inequality (5.12), then, at the same time, we will prove that $\phi(p, t)$ intersects plane $x = x_1$ for $t = t_1$ and also relation (5.8).

We will thus prove inequality (5.12). If $z(p) > x(p)$ and if path $\phi(p, t)$ does not intersect plane $z - x = 0$ until the intersection with plane $x = x_1$, then inequality (5.12) follows immediately because y increases for $z - x > 0$ along all motions of system (2.15). Now suppose that $z(p) - x(p) \leq 0$ or that $z(p) - x(p) > 0$, but that path $\phi(p, t)$ intersects plane $z - x = 0$ before the intersection with plane $x = x_1$.

We will prove inequality (5.12) by contradiction. Suppose that a $t^* > 0$ exists such that

$$y(t^*) = 0.9y(p), \tag{5.13}$$

$$x(t^*) \leq x_1 \tag{5.14}$$

and that inequality (5.12) is fulfilled for $t \epsilon [0, t^*)$; i.e., that t^* is the first point at which inequality (5.12) is violated. Return to the equality

$$\frac{dz}{dx} = -\frac{cx + b\alpha(x)}{y - f(x)}. \tag{5.15}$$

From this equality and inequalities (5.10) and (5.12) it follows that on path $\phi(p, t)$ for $t \epsilon [0, t^*)$ there is fulfilled the inequality

$$\frac{dz}{dx} > -\frac{cx + b\alpha(x)}{89m}.$$

But, by hypothesis of the cases considered, $b < 1$; therefore, $cx + b\alpha(x) < cx + \alpha(x) = f(x)$. Thus from the preceding inequality, we see that

$$\frac{dz}{dx} > -\frac{1}{89}. \tag{5.16}$$

Integrating the last inequality along path $\phi(p, t)$ from 0 to t^*, by virtue of (5.14) we obtain

$$z(0) - z(t^*) < \frac{1}{89} x_1. \tag{5.17}$$

Consider now equality (3.7). Because of this equality, inequalities (5.12) and (5.17) and because $z(p) = z(0) \geq - x_0$, there results

$$\frac{dy}{dx} > -\frac{x_0 + \frac{1}{89} x_1 + x_1}{89m} > -\frac{1}{20}.$$

Integrating this inequality along $\phi(p, t)$ on interval $0 < t < t^*$, we obtain

$$y(p) - y(t^*) < \frac{1}{20} x_1. \tag{5.18}$$

Since $x_1 < m$, the last inequality contradicts equality (5.13). The contradiction obtained thus proves inequality (5.12).

We will now prove inequality (5.9). We have

$$v(t_1) - v(0) = \int_{x(p)}^{x_1} \frac{dv}{dx}\,dx = \int_{x(p)}^{x_0} \frac{dv}{dx}\,dx + \int_{x_0}^{x_1} \frac{dv}{dx}\,dx.$$

Thus, from (4.99) we see that /81

$$v(t_1) - v(0) = -(1-b)\int_{x(p)}^{x_0} \alpha'(x)(y - f(x))\,dx$$
$$-(1-b)\int_{x(p)}^{x_0} \alpha'(x)(y - f(x))\,dx. \tag{5.19}$$

Evaluate the integral standing on the right of this equality. To do so, first evaluate y(t) on interval $0 \leq t \leq t_1$. From equation (3.7), we obtain

$$\frac{dy}{dx} < \frac{|z - x|}{y - m}.$$

Since z decreases with increasing time along path $\phi(p, t)$ for $t \in [0, t_1]$, and since inequality (5.12) is fulfilled for $t \in [0, t_1]$ on $\phi(p, t)$, then from the last inequality and from the condition of case I it follows that

$$\frac{dy}{dx} < 2.$$

Integrating this inequality and using inequality (5.10), we obtain

$$y(t) < 2y(p) \text{ for } t \in [0, t_1].$$

Taking this inequality into account, we can write

$$\int_{x(p)}^{x_0} \alpha'(x)(y - f(x))\,dx > -2Dy(p)\,x_0. \tag{5.20}$$

Moreover, from inequalities (5.10) and (5.12) we obtain

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\,dx > \frac{1}{2}\varepsilon y(p)(x_1 - x_0).$$

Thus, from (5.3) we conclude that

$$\int_{x_0}^{x_1} \alpha'(x)\,(y-f(x))\,dx > 5Dy(p)\,x_0. \tag{5.21}$$

From inequalities (5.19)–(5.21), we obtain inequality (5.9).

We consider now case II. As in case I, it is easy to establish the inequality

$$z(p) > \frac{10^{7}m^2}{x_0}. \tag{5.22}$$

In this case we will show that path $\phi(p, t)$ first intersects plane $x = x_1$ and then plane $z - x = 0$. Obviously, this will be proved by the existence of instant t_1 and inequality (5.8).

Contrary to our assertion, suppose that there exists a $t^* > 0$ such that

$$z(t^*) - x(t^*) = 0, \tag{5.23}$$

$$x(t^*) \leq x_1, \tag{5.24}$$

and for $t \epsilon [0, t^*)$ there is fulfilled the inequality

$$z(t) - x(t) > 0. \tag{5.25}$$

Thus, t^* is the first instant after $t = 0$ of the violation of inequality (5.25); inequality (5.25) for $t = 0$ is fulfilled as proceeds from (5.22). In consequence of equalities (5.23) and (5.22), there exists a $t^{**} \epsilon (0, t^*)$ such that

$$z(0) - z(t^{**}) = m. \tag{5.26}$$

Return to equality

$$\frac{dy}{dz} = -\frac{z-x}{cx + b\alpha(x)}. \tag{5.27}$$

From equalities (5.26) and (5.27) and from inequality (5.22), it follows that on path $\phi(p, t)$ for $t \epsilon (0, t^{**})$ there is fulfilled the inequality

$$\frac{dy}{dz} < -100.$$

Integrating this inequality along path $\phi(p, t)$ from $t = 0$ to $t = t^{**}$, because of (5.26) we obtain

$$y(t^{**}) > 100m. \tag{5.28}$$

Since inequality (5.25) is fulfilled on interval $t^{**} < t < t^*$, then on this interval, $y(t)$ increases, and, consequently, from the last inequality we obtain

$$y(t) > 100m \text{ for } t \in [t^{**}, t^*]. \tag{5.29}$$

From equality (5.15) and inequality (5.29) we conclude that on path $\phi(p, t)$ for $t \in [t^{**}, t^*]$ there is fulfilled the inequality

$$\frac{dz}{dx} > -\frac{1}{99}.$$

By integrating this inequality along path $\phi(p, t)$ for $t^{**} \leq t \leq t^*$ and by using (5.24), we obtain

$$z(t^{**}) - z(t^*) < \frac{1}{99} x_1.$$

Therefore, from (5.26) we obtain

$$z(0) - z(t^*) < m + \frac{1}{99} x_1.$$

The last inequality contradicts equality (5.23) and inequalities (5.22) and (5.24). The contradiction obtained thus proves that path $\phi(p, t)$, in the case considered, first intersects plane $x = x_1$ and then plane $z - x = 0$.

We introduce the following notations. As earlier, let t_0 and t_1 be the instants of intersection of $\phi(p, t)$ with planes $x = x_0$ and $x = x_1$ respectively, and let $t = t'$ be the instant of intersection of $\phi(p, t)$ with plane $x = 2x_0$. Obviously, $0 \leq t_0 < t' < t_1$. For $t \in [t', t_1]$, we will prove that there is fulfilled the inequality

$$y(t) > 10m. \tag{5.30}$$

Indeed, if there exists a $t^{**} \in [0, t']$ such that equality (5.26) is fulfilled for $t = t^{**}$, then, as earlier, we will prove inequality (5.28). From it, also, (5.30) will follow since $y(t)$ for $t \in [0, t_1]$ increases along with time. But if there exists no such t^{**}, then for $t \in [0, t']$ the following inequality is true:

$$z(0) - z(t) < m. \tag{5.31}$$

In this case, we will prove inequality (5.30) by contradiction. Suppose that for $t \in [0, t']$ the following inequality is fulfilled:

$$y(t) \leq 10m. \tag{5.32}$$

Then, from equality (3.7) and inequalities (5.22) and (5.31), there results

$$\frac{dy}{dx} > \frac{100 \frac{m^2}{x_0}}{y},$$

and thus, from (5.32) it follows that

$$\frac{dy}{dx} > 10\frac{m}{x_0}$$

on path $\phi(p, t)$ for $t \in [0, t']$. Integrating the last inequality along $\phi(p, t)$ from $t = 0$ to $t = t'$, we obtain

$$y(t') - y(0) > 10m.$$

Since, by hypothesis, $y(0) \geq 0$, the last inequality contradicts inequality (5.32). The contradiction obtained thus proves inequality (5.30).

Now we will prove inequality (5.9). To do this, we return to equality (5.19). Evaluate the integral standing on the right of this equality

$$\int_{x(p)}^{x_0} \alpha'(x)(y - f(x))\,dx > -Dy(t')\,x_0, \tag{5.33}$$

since $y(t)$ increases for $t \in [0, t_1]$;

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\,dx > \varepsilon(y(t') - m)(x_1 - 2x_0).$$

Therefore, from (5.30) and (5.3) it follows that

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\,dx > 5Dy(t')\,x_0. \tag{5.34}$$

The relation (5.9) also proceeds from (5.19), (5.33) and (5.34). Thus, the lemma is proved.

Lemma 5.2

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Moreover, let condition (5.1) be fulfilled. Suppose that point p lies in one of the domains $\{0 \leq x \leq x_0,\ z \geq 0,\ y = f(x),\ v \geq \frac{1}{2}R^2\}$ or

$$\left\{x = 0,\ y < 0,\ z > 0,\ \frac{y}{z} \geqslant -1,\ v \geqslant \frac{1}{2}R^2\right\},$$

where, as in lemma 5.1, v is the function defined by equality (4.98). Then path $\phi(p, t)$ of system (2.15) intersects plane $x = x_1$ for $t = t_1 < 0$ (by t_1 is understood the first instant after $t = 0$ of the intersection of $\phi(p, t)$ with plane $x = x_1$ in the direction of decreasing time). On path $\phi(p, t)$ there is fulfilled the relations

$$y < f(x) \text{ for } t \in [t_1, 0), \tag{5.35}$$

$$v(p) < v(\phi(p, t_1)). \tag{5.36}$$

Proof

For the proof of this lemma we will consider only one path of system (2.15), the path $\phi(p, t)$. Therefore, as for the proof of lemma 5.1, we will write the different functions of the coordinates of the phase space as functions of time. Also, as for the proof of lemma 5.1, it is easy to establish the inequality

$$z(p) > 105 \frac{m^2}{x_0}. \tag{5.37}$$

We will prove that path $\phi(p, t)$ intersects plane $x = x_1$ for $t = t_1 < 0$. For $t < 0$ and sufficiently close to zero, path $\phi(p, t)$ lies in domain $\{x > 0, y < f(x), z > x\}$, as is easily seen. Path $\phi(p, t)$ can leave this domain only through the plane $z - x = 0$. However, z increases with decreasing time in domain $\{x > 0\}$. Thus, from inequality (5.37) it follows that $\phi(p, t)$ can intersect plane $z - x = 0$ (in the direction of decreasing time) only after the intersection with plane $x = x_1$.

Now suppose that path $\phi(p, t)$ does not intersect plane $x = x_1$ for $t < 0$. Then, obviously, for all $t < 0$, path $\phi(p, t)$ lies in domain $\{0 < x < x_1, y < f(x), z - x > 0\}$.

Consider the following function of the coordinates of the phase space

$$w = \frac{1}{2} y^2 + \frac{1}{2} (z - x)^2. \tag{5.38}$$

The time derivative of this function, taken because of the differential equations of system (2.15), as is easily verified, is equal to

$$\dot{w} = (1 - b)(z - x)\alpha(x). \tag{5.39}$$

From equality (5.39) it follows that function w in domain $\{x > 0, z - x > 0\}$ decreases with time along all motions of system (2.15). Therefore, it results that path $\phi(p, t)$ in domain $\{0 < x < x_1, y < f(x), z - x > 0\}$ is bounded for $t < 0$ and, consequently, for $t \to -\infty$ must go to an equilibrium position different from the origin. But system (2.15) has only one equilibrium position, the point $x = y = z = 0$. The contradiction obtained thus proves that path $\phi(p, t)$ for $t = t_1 < 0$ intersects plane $x = x_1$. In passing, we also established inequality (5.35).

Go now to the proof of inequality (5.36). Let t_0 and t' be the instants of intersection of trajectory $\phi(p, t)$ with planes $x = x_0$ and $x = 2x_0$ respectively. It is clear that $0 \geq t_0 > t' > t_1$. We will prove that

$$y(t') < -9m. \tag{5.40}$$

Suppose to the contrary that

$$y(t') \geq -9m.$$

Since y(t) decreases with decreasing time, from the last inequality there follows the relation

$$y(t) \geq -9m \text{ for } t' \leq t \leq 0. \tag{5.41}$$

From equality (3.7) and inequalities (5.37) and (5.41) follows inequality

$$\frac{dy}{dx} < -10\frac{m}{x_0}$$

on path $\phi(p, t)$ for $0 \geq t \geq t'$.

Integrating the last inequality along path $\phi(p, t)$ from $t = 0$ to $t = t'$, we obtain

$$y(t') - y(0) < -10m. \tag{5.42}$$

But by hypothesis, $y(0) = y(p) \leq f(x(p)) < m$. Therefore, from (5.42) there follows the inequality

$$y(t') < -9m.$$

This inequality contradicts inequality (5.41). The contradiction obtained thus proves inequality (5.40).

Now we will prove inequality (5.36) by returning to equality (5.19). Evaluate the integral standing on the right hand side of this equality,

$$\int_{x(p)}^{x_0} \alpha'(x)\,(y - f(x))\,dx < -D\,(y\,(t') - m)\,x_0, \tag{5.43}$$

since for $t \epsilon [t_1, 0]$, y(t) increases along with time

$$\int_{x_0}^{x_1} \alpha'(x)\,(y - f(x))\,dx < \varepsilon y\,(t')\,(x_1 - 2x_0).$$

Thus, from (5.3) we find that

$$\int_{x_0}^{x_1} \alpha'(x)\,(y - f(x))\,dx < 10Dy\,(t')\,x_0. \tag{5.44}$$

From inequalities (5.39), (5.43) and (5.44) we obtain

$$\int_{x(p)}^{x_1} \alpha'(x)\,(y-f(x))\,dx < 0. \tag{5.45}$$

And, (5.36) also proceeds from (5.19).

Thus the lemma is proved.

Lemma 5.3

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Moreover, let condition (5.1) be fulfilled. Suppose that point p lies in domain $\left\{x = 0,\ y < 0,\ \frac{|z|}{|y|} \leqslant 1,\ v \geqslant \frac{1}{2}R^2\right\}$ where, as before, v is the function defined by equality (4.98). Then path $\phi(p, t)$ of system (2.15) intersects plane $x = x_1$ for $t = t_1$ (t_1 is understood as the first instant after $t = 0$ of the intersection of $\phi(p, t)$ with the plane $x = x_1$ in the direction of decreasing time). And on path $\phi(p, t)$ there are fulfilled the relations

$$y < f(x) \text{ for } t \in [t_1, 0) \tag{5.46}$$

and

$$v(p) < v(\phi(p, t_1)). \tag{5.47}$$

Proof

As earlier, to prove the lemma, we will consider only trajectory $\phi(p, t)$ of system (2.15). In connection with this, write the different functions of the coordinates of the phase space as functions of time. As in the proof of the preceding lemma, it is not difficult to establish the inequality

$$y(p) < -100m. \tag{5.48}$$

Begin moving along path $\phi(p, t)$ in the direction of decreasing time from point p. We will show that until $x \leq x_1$ on path $\phi(p, t)$, the following inequality is fulfilled on it:

$$y < 0.9y(p). \tag{5.49}$$

We will prove this inequality by contradiction. Suppose that, for the motion along the path $\phi(p, t)$ from point p in the direction of decreasing time, inequality (5.49) is violated before inequality $x \leq x_1$ is true. Because inequality (5.49) is fulfilled for $t = 0$, there exists a $t^* < 0$ for which

$$y(t^*) = 0.9y(p), \tag{5.50}$$

$$x(t^*) \leq x_1 \tag{5.51}$$

or

$$y(t) < 0.9y(p) \text{ for } t \in (t^*, 0], \tag{5.52}$$

$$x(t) \leq x_1 \text{ for } t \in [t^*, 0]. \tag{5.53}$$

Return now to equality (3.7). From this equality and inequalities (5.52) and (5.48), and from the fact that $z(t)$ increases with decreasing time for $t \in [t^*, 0]$, it follows that on path $\phi(p, t)$ for $t \in [t^*, 0]$ there is fulfilled the inequality

$$\frac{dy}{dx} < 2.$$

Integrating this inequality along $\phi(p, t)$ from $t = 0$ to $t = t^*$ and using inequalities (5.51) and (5.53), we obtain

$$y(t^*) - y(0) < 2x_1. \tag{5.54}$$

Inequalities (5.48) and (5.54) contradict equality (5.50). The contradiction obtained thus proves the correctness of inequality (5.49).

Now we will prove that path $\phi(p, t)$ for $t = t_1 < 0$ intersects plane $x = x_1$. Assume to the contrary that this is not so. Then for all $t < 0$, path $\phi(p, t)$ lies in domain $\{0 < x < x_1, y < 0.9y(p)\}$. We will show that $\phi(p, t)$ is then bounded for $t < 0$. If $\phi(p, t)$ lies in domain $\{z - x > 0\}$ for all $t < 0$, then the boundedness of path $\phi(p, t)$ for $t < 0$ is proved in the same way as for the proof of lemma 5.2, i.e., by considering the function w introduced by equality (5.38). But let there exist, even if only one, an instant of time $t' < 0$ such that point $\phi(p, t')$ lies in domain $\{0 \leq x \leq x_1, y < 0.9y(p), z - x \leq 0\}$. Then path $\phi(p, t)$ lies in this domain for all $t < t'$. Indeed, it can leave this domain only through the plane $z - x = P$. But, as inequality (3.2) shows, for $x > 0$ and $y < 0$, all motions of system (2.15) intersect plane $z - x = 0$, going with decreasing time from domain $\{z - x > 0\}$ into domain $\{z - x < 0\}$. Therefore, path $\phi(p, t)$ for $t < t'$ lies in domain $\{0 < x \leq x_1, y < 0.9y(p), z - x < 0\}$. In this domain $z(t)$ increases with decreasing time and is bounded below by the number x_1, and $y(t)$ increases and is negative. Thus, path $\phi(p, t)$ for $t < 0$ is bounded and, for sufficiently small t, completely lies in either domain $\{0 < x \leq x_1, y < f(x), z > x\}$ or domain $\{0 < x \leq x_1, y < f(x), z - x < 0\}$. Consequently, for $t \to -\infty$, path $\phi(p, t)$ goes to an equilibrium position of system (2.15) different from point $x = y = z = 0$. This contradicts the fact that system (2.15) has only one equilibrium position. But the contradiction obtained proves that $\phi(p, t)$ for $t = t_1 < 0$ intersects plane $x = x_1$. From inequality (5.49), inequality (5.46) follows.

Now, evaluate $y(t)$ on the interval $t_1 \leq t \leq 0$ from below. From equality (5.15) and inequalities (5.48) and (5.49), we see that

$$\frac{dz}{dx} < \frac{1}{90}$$

on path $\phi(p, t)$ for $t \in [t_1, 0]$. Integrating this inequality along path $\phi(p, t)$ from $t = 0$ to $t = t_1$, we obtain

$$z(t_1) - z(0) < \frac{1}{90} x_1. \tag{5.55}$$

Owing to equality (3.7), inequalities (5.48), (5.49) and (5.55), and the condition that $\frac{|z(p)|}{|y(p)|} \leq 1$, we see that on path $\phi(p, t)$ for $t \in [t_1, 0]$ there is fulfilled the inequality

$$\frac{dy}{dx} > -2.$$

Integrating this inequality along $\phi(p, t)$ from $t = 0$ to $t = t_1$, we obtain

$$y(t_1) - y(0) > -2x_1.$$

Thus, it is clear that

$$y(t_1) > 2 \cdot y(0). \tag{5.56}$$

Now we will prove inequality (5.47). Evaluate the integral standing on the right side of equality (5.19). Derived from inequality (5.56),

$$\int_0^{x_0} \alpha'(x)(y - f(x))\, dx < -D(2y(p) - m)\, x_0. \tag{5.57}$$

From inequality (5.49) follows

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\, dx < 0.9\varepsilon y(p)(x_1 - x_0).$$

Therefore, from (5.3), it results

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\, dx < 9Dy(p)\, x_0 \tag{5.58}$$

From inequalities (5.48), (5.57) and (5.58) proceeds the relation

$$\int_0^{x_1} \alpha'(x)(y - f(x))\, dx < 0. \tag{5.59}$$

From inequality (5.59) and equality (5.19) follows inequality (5.47).

Thus the lemma is proved.

Lemma 5.4

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Moreover, let condition (5.1) be fulfilled. Suppose that point p lies in domain $\left\{x = 0,\ y < 0,\ z < 0,\ \frac{y}{z} \leq 1,\ v \geq \frac{1}{2}R^2\right\}$ where, as before, v is a function of the coordinates of the phase space introduced by equality (4.98). Suppose further that there exists a $t_2 < 0$ such that on trajectory $\phi(p, t)$ there is fulfilled the relations

$$x(t_2) = x_2, \tag{5.60}$$

$$y < f(x) \text{ for } t < \epsilon[t_2, 0]. \tag{5.61}$$

Then path $\phi(p, t)$ intersects plane $x = x_1$ for $t = t_1 \epsilon (t_2, 0)$, and there occurs the following inequality:

$$v(p) < v(\phi(p, t_1)). \tag{5.62}$$

Proof

Being interested in the proof for only path $\phi(p, t)$ of system (2.15), as before we will consider functions of the coordinates of the phase space simply as functions of time. Assume that $\xi = 9x_1$. Because of conditions (5.60) and (5.61), we can assert that there exist instants of time t_0, t_1 and t' such that

$$x(t_0) = x_0,\ x(t_1) = x_1,\ x(t') = \xi, \tag{5.63}$$

and these instants are solely on the interval of time $(t_2, 0)$. Obviously, $t_2 < t' < t_1 < t_0 < 0$. From the conditions that $v \geq \frac{1}{2}R^2$ and $\frac{y(p)}{z(p)} \leq 1$, and from notation (5.6) it is easy to see inequality

$$z(p) < -100\frac{m^2}{x_0}. \tag{5.64}$$

We will prove that on interval $t\epsilon[t_2, 0]$ there is fulfilled the inequality

$$z(t) < 0.9z(0). \tag{5.65}$$

We will prove this inequality by contradiction. Suppose that there exists a $t^* \epsilon [t_2, 0]$ such that

$$z(t^*) = 0.9z(0) \tag{5.66}$$

and that inequality (5.65) is fulfilled for $t\epsilon(t^*, 0]$. From inequality (5.65) we see that $z(t) < 0$ for $t\epsilon[t^*, 0]$, and as the second equation of system (2.15) proves equality (5.39), y(t) and w(t) increase with decreasing time for $t\epsilon[t^*, 0]$. Therefore, we have

$$w(0) \leq w(t^*).$$

From the same form of function w, we obtain

$$z^2(0) + y^2(0) \leq [z(t^*) - x(t^*)]^2 + y^2(t^*). \tag{5.67}$$

But because y(t) increases with decreasing time for $t \epsilon [t^*, 0]$, and, moreover, because of condition (5.61), we see that $y(t) < m$ for $t \epsilon [t^*, 0]$. Therefore, from inequality (5.67) follows relation

$$z^2(0) \leq [z(t^*) - m]^2 + m^2.$$

Thus, from equality (5.66) we obtain

$$z^2(0) \leq 0.81 z^2(0) - 1.8 z(0) m + 2m^2. \tag{5.68}$$

This inequality contradicts inequality (5.64), and the contradiction obtained thus proves inequality (5.65).

Now we will prove that

$$y(t) < -8m \tag{5.69}$$

for $t \epsilon [t', 0]$. Since y(t) decreases with increasing time for $t \epsilon [t_2, 0]$, for the proof of inequality (5.69), it is sufficient to establish that

$$y(t') < -8m. \tag{5.70}$$

We will prove this inequality. Suppose that it is not fulfilled; then for $t \epsilon [t_2, t']$ there is fulfilled

$$y(t) \geq -8m. \tag{5.71}$$

In consequence of equality (3.7) and inequalities (5.64), (5.65) and (5.71), on path $\phi(p, t)$ for $t \epsilon [t_2, t']$ there is fulfilled inequality

$$\frac{dy}{dx} > \frac{90 \frac{m^2}{x_0}}{9m} = 10 \frac{m}{x_0}.$$

Integrating this inequality along $\phi(p, t)$ from $t = t'$ to $t = t_2$, we obtain

$$y(t_2) - y(t') > 10 \frac{m}{x_0} (x_2 - \xi) > 10m.$$

Since, by hypothesis, $y(t_2) \leq f(x(t_2)) \leq m$, from the last inequality we obtain

$$y(t') < -9m.$$

This inequality contradicts inequality (5.71). The contradiction obtained proves inequality (5.70) and, with this, also (5.69).

From equality (3.7) and inequalities (5.64), (5.65) and (5.69), we see that for $t \in [t', 0]$ on $\phi(p, t)$ there is fulfilled inequality

$$\frac{dy}{dx} > \frac{z(p)}{2y}.$$

Multiplying this inequality by $2y < 0$ and integrating the inequality thus obtained along the path $\phi(p, t)$ from $t = 0$ to $t = t'$, we obtain

$$y^2(0) - y^2(t') > -z(p)\xi.$$

Accordingly,

$$y(0) < -\sqrt{-z(p)}\sqrt{\xi}. \tag{5.72}$$

On the other hand, from equality (3.7) and inequalities (5.64), (5.65) and (5.69), we see that for $t \in [t', 0]$ on path $\phi(p, t)$ there is fulfilled inequality

$$\frac{dy}{dx} < \frac{2z(p)}{y}.$$

Multiplying this inequality by $y < 0$ and integrating on the interval $t_1 \leq t \leq 0$, we obtain

$$y^2(0) - y^2(t_1) < -4z(p)x_1$$

or

$$[y(0) - y(t_1)][y(0) + y(t_1)] < -4z(p)x_1.$$

Thus, from (5.72) and (5.69) we obtain

$$y(t_1) - y(0) < \frac{4\sqrt{-z(p)}x_1}{\sqrt{\xi}}.$$

And as a result, from (5.72) we see that

$$y(t_1) - y(0) < -\frac{4y(0)}{\xi}x_1.$$

Since $\xi = 9x_1$, from the last inequality we obtain

$$y(t_1) < \frac{1}{2}y(0). \tag{5.73}$$

Now we will prove inequality (5.62). To do this, we return to equality (5.19) and evaluate the integral standing on the right side of this equality,

$$\int_0^{x_0} \alpha'(x)(y - f(x))\,dx < -D[y(0) - m]\,x_0. \tag{5.74}$$

On the other hand,

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\,dx < \varepsilon\, y(t_1)(x_1 - x_0).$$

Thus, from (5.73) and (5.3), we see that

$$\int_{x_0}^{x_1} \alpha'(x)(y - f(x))\,dx < 5Dy(0)\,x_0\,. \tag{5.75}$$

From inequalities (5.74), (5.75) and (5.69), there follows the inequality

$$\int_0^{x_1} \alpha'(x)(y - f(x))\,dx < 0. \tag{5.76}$$

Inequality (5.62) follows from equality (5.19) and inequality (5.76).

The lemma is proved.

Lemma 5.5

Let inequalities $a > 0$ and $b < 1$ be fulfilled. In addition, let condition (5.1) be fulfilled. Suppose that point p lies in domain $\left\{x = 0,\ y < 0,\ z < 0,\ \frac{y}{z} \le 1,\ v \ge \frac{1}{2}R^2\right\}$ where, as earlier, v is the function introduced by equality (4.98). Suppose also that there exists no $t_2 < 0$ such that relations (5.60) and (5.61) are fulfilled on path $\phi(p, t)$ of system (2.15). Then there exists a $T < 0$ such that on path $\phi(p, t)$ it results that

$$x(T) = 0. \tag{5.77}$$

$$x(t) > 0,\ z(t) < 0 \text{ for } t \epsilon (T, 0). \tag{5.78}$$

Proof

Begin moving along path $\phi(p, t)$ from point p in the direction of decreasing time. We will show that, until points of path $\phi(p, t)$ lie in domain $\{0 \le x \le x_2,\ y \le 10m\}$, on the path is fulfilled inequality

$$z(t) < 0.9z(p). \tag{5.79}$$

Note that by the hypotheses of the lemma, inequality (5.64) is fulfilled. We will prove inequality (5.79) by contradiction. Suppose that there exists a $t^* < 0$ such that

$$z(t^*) = 0.9z(p), \tag{5.80}$$

and for $t \in [t^*, 0]$ it results that $\phi(p, t) \in \{0 \leq x \leq x_2, y \leq 10m, z \leq 0.9z(p)\}$. It is clear that y and w increase with decreasing time along path $\phi(p, t)$ for $t \in [t^*, 0]$. Therefore, we have

$$w(0) \leq w(t^*).$$

From the form of the function w it follows that

$$z^2(p) + y^2(p) \leq [z(t^*) - x(t^*)]^2 + y^2(t^*). \tag{5.81}$$

But for $t \in [t^*, 0]$ along $\phi(p, t)$, y increases with decreasing time. Moreover, $y \leq 10m$ by hypothesis; therefore, from inequality (5.81) we obtain

$$z^2(p) \leq [z(t^*) - m]^2 + 100m^2.$$

And, accordingly, from (5.80) we obtain

$$z^2(p) \leq 0.81z^2(p) - 1.8z(p)m + 101m^2.$$

This inequality contradicts inequality (5.64); the contradiction obtained thus proves inequality (5.79).

We will show that there exists instants of time $t = t_1 < 0$ such that on $\phi(p, t)$ there results

$$y(t_1) = f(x(t_1)) \tag{5.82}$$

and for $t \in (t_1, 0]$, $\phi(p, t) \in \{0 \leq x \leq x_2, y < f(x)\}$.

Indeed, in domain $\{0 \leq x \leq x_2, y < f(x)\}$ along $\phi(p, t)$, y increases with decreasing time due to (5.79); z also increases and, because of (5.79), is bounded. Therefore, path $\phi(p, t)$ goes into domain $\{0 \leq x \leq x_2, y < f(x)\}$ for decreasing time. But it cannot intersect plane $x = x_2$ for $y < f(x)$, for the instant of intersection $t = t_2$ would satisfy relations (5.60) and (5.61). Therefore, path $\phi(p, t)$ intersects surface $y - f(x) = 0$ for $t = t_1$, and $\phi(p, t) \in \{0 \leq x \leq x_2, y < f(x)\}$ for $t \in (t_1, 0]$.

It is easy to see that path $\phi(p, t)$ goes into domain $\{0 \leq x \leq x_2, f(x) < y \leq 10m\}$ for $t < t_2$. If for this case $\phi(p, t)$ intersects plane $x = 0$, then the instant of intersection $t = T < t_1$, because of inequality (5.79), satisfies relations (5.77) and (5.78). Suppose that for $t = \tau < t_1$, $\phi(p, t)$ intersects plane $y = 10m$, and for this, on the path, it results that $x(\tau) \in (0, x_2)$. In consequence of equality (5.15), until the points of path $\phi(p, t)$ lie in domain $\{0 \leq x \leq x_2, y \geq 10m\}$, on it is fulfilled the inequality

$$\frac{dz}{dx} > -0.2.$$

Therefore, for that $t < \tau$ for which $\phi(p, t)$ lies in domain $\{x \geq 0\}$, on it is fulfilled

$$z(t) < 0.8z(p). \tag{5.83}$$

From equality (3.7) it results that for $t < \tau$ for which path $\phi(p, t)$ lies in domain $\{x \geq 0\}$, there is fulfilled inequality

$$\frac{dy}{dx} > \frac{z(p) - m}{9m}.$$

Thus, it follows that path $\phi(p, t)$ is bounded for that t. Consequently, it enters domain $\{x > 0\}$ for $t < \tau$. Accordingly, there exists an instant of time $T < 0$ which satisfies relations (5.77) and (5.78).

The lemma is proved.

Theorem 5.1

Let inequalities $a > 0$ and $b < 1$ be fulfilled. In addition, let condition (5.1) be fulfilled. Let point p lie in plane $x = 0$. Suppose that there is fulfilled inequality

$$y^2(p) + z^2(p) < R^2. \tag{5.84}$$

Moreover, suppose that point $\phi(p, T)$ lies in plane $x = 0$ where $T > 0$. Then the following inequality is true:

$$y^2(\phi(p, T)) + z^2(\phi(p, T)) < R^2. \tag{5.85}$$

Proof

For the proof of this theorem, we will consider only path $\phi(p, t)$ of system (2.15). In connection with this, as earlier, the different functions of points of the phase space will always be considered as functions of time. For definiteness, we will say that $y(p) \geq 0$. In this case, if $z(p) \leq 0$, then we will say that $y(p) \geq 0$. Without losing generality, we can say that T is the first instant after $t = 0$ of the intersection of path $\phi(p, t)$ with plane $x = 0$, i.e., that $x(t) > 0$ for $t \epsilon (0, T)$.

Consider first the case when $z(p) \leq 0$. In consequence of the third equation of system (2.15) and the GHC, $\frac{dz}{dt} < 0$ for $t \epsilon (0, T)$; therefore, $z(t) < 0$ for $t \epsilon (0, T]$, and, consequently, function w, introduced by equality (5.38), decreases along $\phi(p, t)$ for $t \epsilon (0, T)$. Thus, we have

$$w(T) < w(0). \tag{5.86}$$

Inequality (5.85) also follows from the definition of the function w and from (5.86).

Now let $z(p) > 0$. Suppose first that on the interval of time $0 < t < T$, path $\phi(p, t)$ does not intersect a part of surface $\{y = f(x),\ 0 < x \leq x_0,\ z - x \geq 0\}$, going out of domain $\{y - f(x) < 0\}$ into domain $\{y - f(x) > 0\}$. As proceeds from lemma 3.3, path $\phi(p, t)$ for $t = \tau \epsilon (0, T)$ intersects surface $y - f(x) = 0$, going for $t = \tau$ from domain $\{y - f(x) > 0\}$ into domain $\{y - f(x) < 0\}$. Suppose that $x(\tau) \leq x_0$. Then path $\phi(p, t)$ for $t \epsilon (0, T)$

intersects surface $y - f(x) = 0$, going from domain $\{y - f(x) > 0\}$ into domain $\{y - f(x) < 0\}$ only at one time for $t = \tau$. Indeed, path $\phi(p, t)$ can go across from domain $\{y - f(x) > 0\}$ into domain $\{y - f(x) < 0\}$ only with an intersection with a part of surface $\{y = f(x),\ 0 < x \leq x_0,\ z - x \geq 0\}$, this, by hypothesis, is impossible. Thus, for $t \epsilon (0, T)$, $x(t)$ has only one maximum, which occurs for $t = \tau$; consequently, $x(t) \leq x_0$ for $t \epsilon [0, T]$.

We will prove that inequality (5.85) is fulfilled in the case considered. But suppose that inequality (5.85) is not fulfilled; then there is fulfilled the inequality

$$v(T) \geq \frac{1}{2}R^2, \tag{5.87}$$

where v is a function of the points of the phase space introduced by equality (4.98). If $z(T) \geq 0$ or if $z(T) < 0$ and $\dfrac{z(T)}{y(T)} \leq 1$, in consequence of equality (5.87) and lemmas 5.2 and 5.3, path $\phi(p, t)$ intersects plane $x = x_1$ for $t \epsilon (0, T)$, which contradicts inequality $x(t) \leq x_0$ for $0 \leq t \leq T$. But if $z(t) < 0$, $\dfrac{y(T)}{z(T)} \leq 1$, and path $\phi(p, t)$ for $t \epsilon (0, T)$ does not intersect plane $x = x_2$, then because of lemma 5.5, it must be true that $z(0) = z(p) \leq 0$; this contradicts supposition $z(p) > 0$. The contradictions obtained thus prove inequality (5.85) in the case considered.

Now let $x(\tau) > x_0$. Then path $\phi(p, t)$ intersects plane $x = x_0$ for $t = t_0 \epsilon (0, T)$. By $t = t_0$ is understood the first instant after $t = 0$ of the intersection of $\phi(p, t)$ with plane $x = x_0$. Let $t = t_3$ be the last instant before $t = T$ that path $\phi(p, t)$ intersects with plane $x = x_0$. From the condition that path $\phi(p, t)$ does not intersect a part of the surface $\{y = f(x),\ 0 < x \leq x_0,\ z - x \geq 0\}$, going from domain $\{y - f(x) < 0\}$ into domain $\{y - f(x) > 0\}$, in this case it follows that path $\phi(p, t)$ lies wholly inside the half-space $\{x \geq x_0\}$ for $t \epsilon [t_0, t_3]$ and in the zone $\{0 \leq x \leq x_0\}$ for $t \epsilon [0, t_0]$ and $t \epsilon [t_3, T]$.

We will now prove inequality (5.85) in the case considered. If $v(t) < \frac{1}{2}R^2$ for $t \epsilon [0, T]$, inequality (5.85) follows immediately from the form of function v. Suppose that an instant of time $t = \theta \epsilon [0, T]$ exists such that $v(\theta) = \frac{1}{2}R^2$; for this case we will say that θ is the first such instant, i.e., that $v(t) < \frac{1}{2}R^2$ for $t \epsilon [0, \theta)$. Since

$\dfrac{dv}{dt} < 0$ for $x \geq x_0$, as proceeds from (4.99) and condition (5.1) of the theorem, point $\phi(p, \theta)$ mu in zone $\{0 \leq x \leq x_0\}$. Thus, one of two conditions must be fulfilled: either $\theta \epsilon [0, t_0]$, or $\theta \epsilon [t_3, T]$.

First suppose that $\theta \epsilon [t_3, T]$. Inequality (5.85) will be proved by contradiction. Suppose that it is not fulfilled, i.e.,

$$v(T) \geq \frac{1}{2}R^2. \tag{5.88}$$

From this inequality, from the fact that $z(p) > 0$, and from lemmas 5.2–5.5, $\phi(p, t)$ consequently intersects plane $x = x_1$ for $t = t_2 \epsilon (t_0, t_3)$. Thus there occurs the inequality

$$v(t_1) > \frac{1}{2}R^2. \tag{5.89}$$

However, $t_1 < t_3 \leq \theta$, and by the definition of θ, $v(t) < \frac{1}{2}R^2$ for all $t \epsilon [0, \theta)$. This shows that inequality (5.88) in the case considered also cannot be realized. Consequently, inequality (5.85) is fulfilled in the case considered.

Now let $\theta \epsilon [0, t_0]$. We will show that in this case

$$z(\theta) > -x_0. \tag{5.90}$$

Suppose to the contrary that

$$z(\theta) \leq -x_0. \tag{5.91}$$

We will show that then

$$z(0) \leq 0. \tag{5.92}$$

Consider first the case $\frac{z(\theta)}{y(\theta)} \leq 1$. Consequently, by definition of the instant and from the form of function v,

$$y(\theta) > 100m. \tag{5.93}$$

Assume that there exists a $t^* \epsilon [0, \theta)$ such that

$$z(t^*) = 0 \tag{5.94}$$

and

$$z(t) < 0 \text{ for } t \epsilon (t^*, \theta]. \tag{5.95}$$

Then y(t) increases with decreasing time for $t \epsilon (t^*, \theta]$. Therefore, from inequality (5.93) and from equation (5.15), it follows that on $\phi(p, t)$ for $t \epsilon [t^*, \theta]$ there is fulfilled the inequality

$$\frac{dz}{dx} > -\frac{1}{99}.$$

Integrating this inequality along path $\phi(p, t)$ from $t = t^*$ to $t = \theta$ and using inequality (5.91), we ascertain that equality (5.94) cannot be realized; consequently, inequality (5.92) must be fulfilled. Now let $\frac{y(\theta)}{|z(\theta)|} \leq 1$. In this case, inequality (5.92) is proved in the same way as for the proof of lemma 5.5. Thus, (5.92) is fulfilled by supposition (5.91). However, inequality (5.92) contradicts the fact that $z(p) > 0$. The contradiction obtained proves inequality (5.90). In consequence of inequality (5.90) and lemma 5.1, path $\phi(p, t)$ intersects plane $x = x_1$ for $t = t_1 \epsilon (t_0, t_3)$ and

$$v(t_1) < v(\theta) = \frac{1}{2}R^2. \tag{5.96}$$

We will now prove inequality (5.85) by contradiction. Suppose that inequality (5.88) is fulfilled. Then, because $z(p) > 0$ and from lemmas 5.2–5.5, we see that path $\phi(p, t)$ intersects plane $x = x_1$ for $t = t_2 \epsilon (t_1, t_3)$. And, consequently,

$$v(t_2) > v(T) \geq \frac{1}{2} R^2. \tag{5.97}$$

But as was proved above for $t \epsilon [t_1, t_2] \subset [t_0, t_3]$, path $\phi(p, t)$ lies in half-space $\{x \geq x_0\}$. However, as equality (4.99) and condition (5.1) of the theorem prove, function $v(t)$ decreases for $x \geq x_0$. Accordingly, it follows that

$$v(t_1) \geq v(t_2).$$

The last inequality contradicts inequality (5.96) and (5.97). The contradictions obtained thus prove inequality (5.85).

Now we consider the case when path $\phi(p, t)$ for $t \epsilon (0, T)$ intersects a part of surface $\{y = f(x),\ 0 < x \leq x_0,\ z - x \geq 0\}$, for this going from domain $\{y - f(x) < 0\}$ into domain $\{y - f(x) > 0\}$. It is not difficult to see that the number of such intersections is finite. Let $\tau_1, \tau_2, \ldots, \tau_k$ be a sequence of instants of such intersections. Then we have

$$0 < \tau_1 < \tau_2 < \ldots < \tau_k < T. \tag{5.98}$$

On the intervals of time $0 < t < \tau_1$, $\tau_1 < t < \tau_2$, $\ldots$, $\tau_{k-1} < t < \tau_k$, $\tau_k < t < T$, path $\phi(p, t)$ does not intersect a part of the surface $\{y = f(x),\ 0 < x \leq x_0,\ z - x \geq 0\}$ in going from domain $\{y - f(x) < 0\}$ into domain $\{y - f(x) > 0\}$. Therefore, by reasoning analogous to that used for the proof of inequality (5.85), we will finally prove inequality

$$v(\tau_i) < \frac{1}{2} R^2 \ (i = 1, 2, \ldots, k). \tag{5.99}$$

And from inequality $v(\tau_k) < \frac{1}{2} R^2$, we conclude inequality $v(T) < \frac{1}{2} R^2$ which coincides with (5.85).

Thus the theorem is proved.

Analogously, the following theorem may be proved.

Theorem 5.2

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Moreover, let condition (5.1) be fulfilled. Let point p lie in plane $x = 0$. Suppose that there is fulfilled inequality

$$y^2(p) + z^2(p) \geq R^2. \tag{5.100}$$

Assume further that point $\phi(p, T)$ lies in plane $x = 0$ where $T > 0$; then there is true the inequality

$$y^2(\phi(p, T)) + z^2(\phi(p, T)) < y^2(p) + z^2(p). \tag{5.101}$$

From the results of reference 12 we see that, in the cases considered, the null solution of system (2.15) is asymptotically stable in the sense of Lyapunov. Therefore, following from (3.30), there exists a domain A of the phase space such that for solutions $x = x(t)$, $y = y(t)$, $z = z(t)$ of system 2.15, relation

$$\lim x = \lim y = \lim z = 0 \text{ for } t \to +\infty \tag{5.102}$$

is fulfilled only when the initial point of this solution lies in domain A. The domain is the domain of stability and, as shown by N. P. Yerugin (ref. 30), is made up of its invariant set (ref. 26). The complement of A relative to the entire phase space is designated by B and is the set of instability. The set B is empty only if the null solution of system (2.15) is globally stable.

The following theorem is true.

Theorem 5.3

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Let condition (5.1) be fulfilled. If the set B is nonempty and if $p \epsilon B$, then path $\phi(p, t)$ of system (2.15) has points in domain $\{x = 0, y^2 + z^2 < R\}$.

Proof

Following from theorem 3.1 there exists a sequence of instants of time

$$t_1 < t_2 < t_3 < \ldots \to +\infty, \tag{5.103}$$

such that points $\phi(p, t_k)$ lie in plane $x = 0$ for all real k.

Contrary to the assertion of the theorem, assume that path $\phi(p, t)$ has no points in domain $\{x = 0, y^2 + z^2 < R^2\}$; then for all real k it is true that

$$y^2(t_k) + z^2(t_k) \geq R^2 \tag{5.104}$$

on path $\phi(p, t)$. Accordingly, by theorem 5.2 we have

$$\begin{gathered} y^2(t_{k+1}) + z^2(t_{k+1}) < y^2(t_k) + z^2(t_k) < \\ < y^2(t_1) + z^2(t_1) \quad \text{for} \quad k > 1. \end{gathered} \tag{5.105}$$

Because of inequality (5.105) and the Bolzano-Weirstrasse principle of choice, we can say that the sequence of points $\phi(p, t_k)$ converges. Assume that

$$\lim_{k \to +\infty} \varphi(p, t_k) = q. \tag{5.106}$$

From inequality (5.104) we obtain

$$y^2(q) + z^2(q) \geq R^2. \tag{5.107}$$

On the other hand, from (5.105) and (5.106), on path $\phi(p, t)$ there is fulfilled inequality

$$y^2(t_k) + z^2(t_k) > y^2(q) + z^2(q) \tag{5.108}$$

for all real k.

Let t^* be an instant of time such that $\phi(p, t^*)$ lies in plane $x = 0$. Then because of theorem 5.2 and relations (5.103), (5.104) and (5.108), on path $\phi(p, t)$ we have

$$y^2(t^*) + z^2(t^*) > y^2(q) + z^2(q). \tag{5.109}$$

Now consider path $\phi(q, t)$ of system (2.15). This path will obviously be an ω-limit for $\phi(p, t)$. Path $\phi(q, t)$ for all $t \geq 0$ cannot lie in one of the half-spaces $\{x \geq 0\}$ or $\{x \leq 0\}$ since, in this case, according to theorem 3.1, path $\phi(q, t)$ would go to origin. This is impossible because p, by assumption, belongs to the set of instability B. Consequently, a $t' > 0$ can be found such that path $\phi(p, t)$ for $t = t'$ crosses over from one of the half-spaces $\{x \geq 0\}$ and $\{x \leq 0\}$ into the other. Due to theorem 5.2 we will have

$$\phi(q, t') \epsilon \{x = 0,\ y^2 + z^2 < y^2(q) + z^2(q)\}. \tag{5.110}$$

By the theorem on integral continuity, a t^* can then be chosen such that also

$$\phi(p, t^*) \epsilon \{x = 0,\ y^2 + z^2 < y^2(q) + z^2(q)\}. \tag{5.111}$$

The last relation contradicts inequality (5.109).

The contradiction obtained thus proves the theorem.

Theorem 5.4

Let inequalities $a > 0$ and $b < 1$ be fulfilled. Moreover, let condition (5.1) be fulfilled. Then there exists such an $M > 0$ that from any point p of the phase space a T_p can be found such that for $t \geq T_p$ on path $\phi(p, t)$ of system (2.15) there are fulfilled the inequalities

$$|x| < M,\ |y| < M,\ |z| < M. \tag{5.112}$$

Proof

Consider an arbitrary path $\phi(p, t)$ of system (2.15). If $\phi(p, t)$ goes to origin for $t \to +\infty$, then the assertion of the theorem is obviously fulfilled for it. However, suppose that $\phi(p, t)$ does not go to origin. Then by theorem 5.3, $\phi(p, t)$ intersects circle $\{x = 0, y^2 + z^2 < R^2\}$ for $t = T_p$. Assume that

$$N = \max |f(x) + x| \text{ for } |x| \leq R. \tag{5.113}$$

We will show that on path $\phi(p, t)$ for $t \geq T_p$ there is fulfilled the inequality

$$|y| < 3N. \tag{5.114}$$

Let $t_1 > T_p$ be the first instant of time after T_p of the intersection of path $\phi(p, t)$ with plane $x = 0$. We will show that on the interval $T_p \leq t \leq t_1$ on $\phi(p, t)$ there is fulfilled inequality (5.114). For definiteness we will say that $y(\phi(p, T_p)) \geq 0$, but if $z(\phi(p, T_p)) \leq 0$, then $y(\phi(p, T_p)) > 0$. If $z(\phi(p, T_p)) \leq 0$, then, as mentioned earlier, y decreases along $\phi(p, t)$ for $t \epsilon [T_p, t_1]$. Consequently, for such t there is fulfilled the inequality

$$|y(\phi(p, t))| \leq \max \{y(\phi(p, T_p)), |y(\phi(p, t_1))|\}.$$

From the last inequality and from theorem 5.1, we see that $|y| < R$ on path $\phi(p, t)$ for $t \epsilon [T_p, t_1]$ and, thus, (5.114) also follows.

Now let $z(\phi(p, T_p)) > 0$. Since $y^2(\phi(p, T_p)) + z^2(\phi(p, T_p)) < R^2$, then $z(\phi(p, T_p)) < R$. The maximum y, as mentioned in section 3, lies on plane $z - x = 0$. But along $\phi(p, t)$, z decreases for $t \epsilon [T_p, t_1]$; therefore, the intersection of $\phi(p, t)$ with plane $z - x = 0$ on interval $t \epsilon [T_p, t_1]$ has an abscissa less than R. We will show that for $t \epsilon [T_p, t_1]$ on path $\phi(p, t)$ there results

$$y < 3N. \tag{5.115}$$

Indeed, if inequality $y \leq 2N$ is fulfilled, inequality (5.115) follows immediately from it. However, if this inequality is not fulfilled, then, as is easily seen, there exists a $t^* \epsilon (T_p, t_1)$ such that

$$y(t^*) = 2N \text{ and } z(t^*) - x(t^*) > 0 \tag{5.116}$$

on path $\phi(p, t)$. Accordingly, until the intersection with plane $z - x = 0$ on path $\phi(p, t)$, inequality $y \geq 2N$ will be fulfilled. Because of inequality, because $z(\phi(p, t)) < R$ for $t \epsilon [T_p, t_1]$ and because of equality (3.7), for $t \geq t^*$, and such that $z(\phi(p, t)) - x(\phi(p, t)) - x(\phi(p, t)) \geq 0$, there is fulfilled the inequality

$$\frac{dy}{dx} = \frac{z - x}{y - f(x)} < 1.$$

Integrating this inequality along $\phi(p, t)$ from $t = t^*$ to the point of intersection of $\phi(p, t)$ with plane $z - x = 0$ and using equality (5.116) and the fact that $x(\phi(p, t)) < R$ lies on plane $z - x = 0$, we obtain inequality (5.115).

Now we will show that for $t \epsilon (T_p, t_1)$ on path $\phi(p, t)$ there is fulfilled inequality

$$y > -3N. \tag{5.117}$$

Since the minimum of y occurs on plane $z - x = 0$, the minimum of this becomes the maximum. From (5.38) and (5.39), between the maximum and the minimum y, function w along $\phi(p, t)$ decreases with increasing time. Consequently, from the form of function w, in the instant of minimum, y is less in absolute value than at the instant of the preceding maximum. Therefore, (5.117) results and from it also (5.115) and (5.114).

Since x decreases along path $\phi(p, t)$ for $y \leq f(x)$, then from the GHC $f(x) > cx$, it results for $x = 0$ that on path $\phi(p, t)$ for $t \epsilon [T_p, t_1]$ there is fulfilled the inequality

$$x < \frac{1}{c} \max y.$$

Thus, from (5.114) it follows that for $t \epsilon [T_p, t_1]$ on path $\phi(p, t)$ there is fulfilled the inequality

$$|x| < \frac{3}{c} N. \tag{5.118}$$

Since z decreases on interval $T_p \leq t \leq t_1$ along $\phi(p, t)$, in consequence of theorem 5.1, on the interval $T_p \leq t \leq t_1$ on $\phi(p, t)$

$$|z| < R. \tag{5.119}$$

Inequalities (5.114), (5.118) and (5.119) are proved only for the interval of time $[T_p, t_1]$, but because of theorem 5.1, obviously, it is also true for all $t \geq T_p$. Therefore, the assertion of the theorem also follows.

Section 17

In this section we consider the case when $a > 0$, $0 \leq b < 1$, $c \geq 1$ and condition (5.1) is fulfilled. We will prove in these cases one theorem relative to the arrangement of the trajectories of system (2.15). Designate domain $\{x = 0, y > 0, z > 0, y^2 + z^2 < R\}$ by P. Consider a periodic motion of system (2.15). As follows from theorems 3.2, 3.3, and 5.3, its trajectory intersects domain P. Call the periodic motion of system (2.15) regular if its trajectory has a point in common with domain P.

Theorem 5.5

Let inequalities $a > 0$, $0 \leq b < 1$ and $c \geq 1$ be fulfilled. Moreover, let function f(x) be continuously differentiable for all real x, and let such positive numbers ϵ and x_0 exist that condition (5.1) is fulfilled. Then, for the trivial solution $x = y = z = 0$ of system (2.15) to be globally stable, it is necessary and sufficient that system (2.15) not have a regular periodic motion.

Proof

The necessity of the theorem's conditions are obvious. We will prove the sufficiency.

Let system (2.15) not have a regular periodic motion. We must then prove that all motions of system (2.15) go to origin for $t \to +\infty$. Suppose to the contrary that there exists a point q of the phase space for which trajectories $\phi(q, t)$ do not go to origin for $t \to +\infty$. Because of theorems 3.2, 3.3, and 5.3, and not violating the generality, we can say that $q \epsilon P$.

Consider now an arbitrary point $p \epsilon \overline{P}$, $p \neq (0, 0, 0)$ and paths $\phi(p, t)$ of system (2.15). Due to theorems 3.2, 3.3 and 5.1, there exists a $t_p > 0$ such that $\phi(p, t_p) \epsilon P$. For this, t_p designates the first instant after $t = 0$ of the intersection of path $\phi(p, t)$ with domain P. We place every point $p \epsilon \overline{P}$, $p \neq (0, 0, 0)$ in correspondence with a point $\phi(p, t_p)$ and point (0, 0, 0) with itself. By I we denote the transformation of the closed domain $\overline{P}$ into itself obtained in this way. From the theorem on uniqueness, the theorem on integral continuity and theorems 3.2, 3.3, and 5.1, it follows that the transformation I is mutually singlevalued and mutually continuous. In addition, it preserves operations. Indeed, we take an arbitrary closed contour l lying in P, and in some way we orient it. We examine surface $\phi(l, t)$ until its intersection with $\overline{P}$ following $t = 0$. This intersection obviously gives us contour I(l). The orientation of contour I(l) cannot coincide with the orientation of contour l only in that case if $\phi(l, t)$ makes an intersection of the trajectory on the surface, which cannot be by virtue of the theorem on uniqueness.

Consider the sequence $I^n(q)$. For all natural n it results that $I^n(q) \epsilon P$. Since domain P is bounded, sequence $I^n(q)$ has a limit point lying in the closed domain $\overline{P}$. Let q_0 be any limit point of sequence $I^n(q)$. In consequence of theorems 3.3 and 5.1,point q_0 coincides with the origin if it lies on the boundary of the domain P. But the point q_0 is a limit for sequence $I^n(q)$; therefore, it is an ω-limit for trajectory $\phi(q, t)$ and, consequently, q_0 cannot coincide with the origin. Indeed, if q_0 were to coincide with point $x = y = z = 0$, path $\phi(p, t)$ would go to origin for $t \to +\infty$ since point (0, 0, 0) is a Lyapunov stable equilibrium position of system (2.15). And this contradicts the choice of point q. Thus, there exists a subsequent $I^n(q)$ of sequence $I^n(q)$ for which is fulfilled the relation

$$\lim_{t \to +\infty} I^{n_k}(p) = q_0 \in P. \tag{5.120}$$

Therefore, we have a homeomorphic and operation-preserving transformation I of plane domain P into itself. This transformation is such that there exists a point $q \epsilon P$ for which relation (5.120) is fulfilled. However, from the theorem of Brouwer (ref. 31), transformation I has a stationary point p_0 lying in domain P (and, consequently, different from the origin). By definition of transformation I, path $\phi(p_0, t)$ shows itself to be a regular periodic motion of system (2.15). This contradicts the hypothesis of the theorem, and the contradiction obtained thus proves the sufficiency of the conditions of the theorem.

We note that Massera (ref. 32) pointed out a possible application of the Brouwer theorem for the investigation of a similar system of differential equations.

Assume again that inequalities $a > 0$, $0 \leq b < 1$, $c \geq 1$ and condition (5.1) are fulfilled.

Suppose that the null solution of system (2.15) is not globally stable. As earlier let A be the stability domain and B the set of instability. Let $p \epsilon \overline{P}$ and $p \neq (0, 0, 0)$; we will, as before, designate by t_p the first instant of time after $t = 0$ in which path $\phi(p, t)$ intersects with domain P.

We will prove that there exists a number T such that $t_p < T$ if $p \epsilon \overline{P} \cdot B$. Assume to the contrary that such T does not exist; then there exists a sequence of points $p_k \epsilon \overline{P} \cdot B$ such that

$$\lim_{k \to +\infty} t_{p_k} = +\infty. \tag{5.121}$$

Since set $\overline{P} \cdot B$ is closed and bounded, we can say that the sequence of points p_k converges to a point of this set; i.e., we can say that

$$\lim_{k \to +\infty} p_k = q \in \overline{P} \cdot B. \tag{5.122}$$

But point (0, 0, 0) obviously lies in domain A; therefore, $q \neq (0, 0, 0)$. On the other hand, because of $q \epsilon \overline{P}$ there exists such an instant of time $t_q > 0$ that $\phi(q, t_q) \epsilon P$. Accordingly, from the theorem on the continuous dependence of the solutions on the initial conditions and from relation (5.122), it follows that for sufficiently large k there takes place the inequality

$$t_{p_k} < 2t_q. \tag{5.123}$$

The last inequality contradicts relation (5.121), and the contradiction obtained proves the existence of number T. Proceeding from the existence of such a number and from theorems 3.2, 3.3 and 5.3, any recurrent (ref. 26) trajectory of system (2.15) different from the equilibrium position produces its vibration regime (the concept of vibration regimes and some conditions for their existence were given by V. V. Nemytskiy in reference 22). Therefore, theorem 5.6 follows.

Theorem 5.6

Let inequalities $a > 0$, $0 \leq b < 1$ and $c \geq 1$ be fulfilled. Moreover, let the function $f(x)$ be continuously differentiable for all real x and let such positive numbers ϵ and x_0 exist that condition (5.1) is fulfilled. Then any trajectory of system (2.15) for $t \to +\infty$ not going to origin has in itself an ω-limit set of vibration regimes.

The assertion of this theorem results because any positively stable trajectory (in the sense of LaGrange trajectories) has in itself an ω-limit set of recurrent motions (refs. 26, 33) and because in the conditions of the theorem any recurrent trajectory of system (2.15) produces its vibration regime.

Chapter VI. ON PERIODIC MOTIONS

In this chapter we will consider system (2.15) for the fulfillment of conditions of cases 1, 4 or 5; i.e., we will assume that $a > 0$ and $b < 1$. Moreover, we will suppose that inequality $c^2 + b > 0$ is fulfilled where number c is given by equality (4.6). Thus, we consider here all of those cases in which the global stability of the null solution of system (2.15) could not be established for any nonlinearity f(x) satisfying the GHC. In these cases we establish certain sufficient conditions for the existence of periodic motions of the solutions for system (2.15).

Section 18

We introduce the following designation:

$$k = \max\left\{1, \frac{1}{c}\right\}. \qquad (6.1)$$

Let H and h be arbitrary numbers satisfying the following inequalities:

$$\frac{1}{c} < h \leq H \text{ for } b \geq 0 \qquad (6.2)$$

and

$$\frac{1}{c} < h \leq H < -\frac{c}{b} \text{ for } b < 0. \qquad (6.3)$$

Since we assume that inequality $c^2 + b > 0$ is fulfilled for the remainder of this chapter, then inequality (6.3) is noncontradictory.

Assume that

$$\lambda = \max\left\{1.1(1-b)cH;\ 1.8\left(c + \frac{b}{c}\right)H\right\} \qquad (6.4)$$

and

$$\mu = \lambda + 0.3c^2 + 0.8 + 0.3\left(c + \frac{b}{c}\right)H. \qquad (6.5)$$

Now let ϵ and δ be arbitrary numbers satisfying the following inequalities:

$$0 < \epsilon - \delta \leq \delta^2, \tag{6.6}$$

$$0 < (c + H)\epsilon < 0.1, \tag{6.7}$$

$$0.625\left(1 + \frac{bH}{c}\right)\delta + 0.35(c + H)c\delta < 0{,}1\left(h - \frac{1}{c}\right), \tag{6.8}$$

$$\mu\varepsilon^2 + 1.8\left(c + \frac{b}{c}\right)H\varepsilon^2 < 0.18\left(h - \frac{1}{c}\right)\delta, \tag{6.9}$$

$$0.08\,ck\frac{H}{k - 0{,}1}\left[\mu + 1.8\left(c + \frac{b}{c}\right)H\right]\varepsilon^2 + \frac{20\,ckH}{3(k - 0.1)}\varepsilon^2$$

$$+ 2\sqrt{k^2(1 + c^2) + 0.012}\,\delta^2 < \frac{0.35}{k}c\left(h - \frac{1}{c}\right)h\delta. \tag{6.10}$$

Suppose that function $\alpha(x)$ satisfies the following conditions:

$$hx \leq \alpha(x) \leq Hx \text{ for } 0 \leq x \leq \delta, \tag{6.11}$$

$$0 < \alpha(x) \leq Hx \text{ for } \delta \leq x \leq \epsilon, \tag{6.12}$$

$$0 < \alpha(x) < \delta^4 \text{ for } \epsilon \leq x \leq \frac{1}{c}\sqrt{k^2(1 + c^2) + 0.012}. \tag{6.13}$$

In these cases, the numbers h, H, δ and ϵ must satisfy inequalities (6.2), (6.3) and (6.6)–(6.10). We will assume that conditions (6.11)–(6.13) are fulfilled.

On plane $x = 0$ consider a point p_0 with coordinates $x = 0$, $y = 0$, $z = ck$; point p_1 with coordinates $x = 0$, $y = k$, $z = ck$; and the segment of the straight line $\{x = 0, z = ck\}$ contained between points p_0 and p_1. Let p_σ be a point of segment L with the y component σk so that $\sigma\epsilon[0.1]$. We will consider a path $\phi(p_\sigma, t)$ of system (2.15). For shortened notation of these trajectories and all functions along them, we will supply σ as an index where $\sigma\epsilon[0.1]$. For example, $\phi_\sigma(t)$ is the trajectory $\phi(p_\sigma, t)$, $y_1(t)$ is the y component of trajectory $\phi(p_1, t)$, and $z_0(t)$ is the z component of the trajectory $\phi(p_0, t)$.

As seen from the reasoning of section 3, all trajectories $\phi_\sigma(t)$ for sufficiently small positive t lie in half-space $x > 0$. Designate by T_σ the first instant after $t = 0$ of the intersection of path $\phi_\sigma(t)$ with plane $x = 0$. In this case, if $x_\sigma(t) > 0$ for all $t > 0$, then we will assume that $T_\sigma = +\infty$. However, in the following, we will show that all paths $\phi_\sigma(t)$ intersect plane $x = 0$ for increasing time from $t = 0$, and thus T_σ turns out to be a proper number.

Lemma 6.1

If conditions (6.11)–(6.13) are fulfilled, then for increasing time from $t = 0$, all paths $\phi_\sigma(t)$ intersect first plane $x = \epsilon$ and then plane $- x = 0$.

For the proof of the lemma we will show that until the intersection with plane $x = \epsilon$, on any trajectory $\phi_\sigma(t)$ there is fulfilled the inequality

$$z_\sigma(t) - x_\sigma(t) > 0.8ck. \tag{6.14}$$

By virtue of lemma 3.9 it is sufficient to establish inequality (6.14) only for trajectory $\phi_0(t)$. Suppose that inequality (6.14) is violated on path $\phi_0(t)$ until its intersection with plane $x = \epsilon$. Then, as long as

$$z_0(0) - x_0(0) = ck > 0.8ck, \tag{6.15}$$

from continuity we can assert that there exists a $t^* > 0$ for which

$$z_0(t^*) - x_0(t^*) = 0.8ck, \tag{6.16}$$

$$z_0(t) - x_0(t) > 0.8ck \text{ for } 0 \leq t < t^*. \tag{6.17}$$

Evaluate $y_0(t^*)$ from the above. We will show that

$$y_0(t) < 0.8 \text{ for } t \epsilon [0, t^*]. \tag{6.18}$$

Indeed, $y_0 = 0$ for $t = 0$; therefore, inequality (6.18) is fulfilled for $t = 0$. Suppose that there exists a $\bar{t} \epsilon (0, t^*)$ for which

$$y_0(\bar{t}) = 0.6, \tag{6.19}$$

and that for $t \epsilon [0, \bar{t}]$ there is fulfilled the inequality

$$y_0(t) \leq 0.6. \tag{6.20}$$

If it results that such a $\bar{t}$ does not exist on interval $(0, t^*)$, due to the continuity of function $y_0(t)$, inequality (6.18) will be fulfilled for all $t \epsilon [0, t^*]$.

Return to equality (3.7). From this equality and from the monotonicity of the increase of function $y_0(t)$ on the interval of time $0 \leq t \leq t^*$, we see that for $t \epsilon [\bar{t}, t^*]$, on path $\phi_0(t)$ there is fulfilled the inequality

$$\frac{dy}{dx} = \frac{z - x}{y - f(x)} < \frac{ck}{0.6 - cx - Hx},$$

since $x_0(t)$ increases monotonically for $t \epsilon [0, t^*]$; then $x_0(t) \leq \epsilon$ for $t \epsilon [0, t^*]$. Therefore, from the last inequality and from condition (6.7),

$$\frac{dy}{dx} < 2ck.$$

Integrating this inequality along path $\phi_0(t)$ for $t \epsilon [\bar{t}, t^*]$, we obtain, because of inequality $x_0(t^*) \leq \epsilon$,

$$y_0(t^*) - y_0(\bar{t}) < 2ck\epsilon. \tag{6.21}$$

But from inequality (6.7) and (6.2) or (6.3), it follows that

$$\left(\frac{1}{c}+c\right)\varepsilon<0{,}1. \tag{6.22}$$

Therefore, $ck\epsilon < 0.1$ since $ck = \max\{1, c\}$ by definition of k. Accordingly, from inequality (6.21) and equality (6.19) we obtain

$$y_0(t^*) < 0.8.$$

Since $y_0(t)$ is monotonically increasing for $t\epsilon[0, t^*]$, the last inequality thus proves relation (6.18).

From equation (3.5) and from condition (6.17), we see that on path $\phi_0(t)$ for $t\epsilon(0, t^*)$ there is fulfilled the inequality

$$\frac{dz}{dy}>-\frac{cx+ba(x)}{0{,}8}.$$

Because $x_0(t)$ increases for $0 \leq t \leq t^*$ and $x_0(t) \leq \epsilon$, from the last inequality we obtain

$$\frac{dz}{dy}>-\frac{c\varepsilon+bH\varepsilon}{0{,}8}.$$

But in this chapter we consider only the case when $b < 1$; therefore, from the last inequality and from condition (6.7) we obtain

$$\frac{dz}{dy}>-\frac{0{,}1}{0{,}8}$$

on path $\phi_0(t)$ for $t\epsilon(0, t^*)$. Integrating this inequality along $\phi_0(t)$ from $t = 0$ to $t = t^*$, we obtain

$$z_0(t^*)-z_0(0)>-\frac{0{,}1}{0{,}8}(y_0(t^*)-y_0(0)).$$

Thus, from relation (6.18) we see that $z_0(t^*) - z_0(0) > -0.1$. But by hypothesis, $z_0 = ck$; therefore, from the last inequality we have

$$z_0(t^*) > 0.9ck. \tag{6.23}$$

From inequality (6.22) it is easy to conclude that inequality $\epsilon \leq 0.05$. Therefore, following from $x(t^*) \leq \epsilon$ and from (6.23),

$$z_0(t^*) - x_0(t^*) > 0.8ck. \tag{6.24}$$

The last inequality contradicts equality (6.16), and the contradiction obtained thus proves the lemma.

In this way, all paths $\phi_\sigma(t)$ intersect first plane $x = \epsilon$ and then plane $z - x = 0$. Designate by $t_\sigma^{(1)}$ the first instant of time after $t = 0$ of the intersection of $\phi_\sigma(t)$ with plane $x = \epsilon$. In going through the proof of the lemma, we have established that for all $t \epsilon [0, t_\sigma^{(1)}]$ there are fulfilled inequality (6.19) and inequality

$$z_\sigma(t) > 0.9ck. \tag{6.25}$$

From the proof of lemma 3.3, it was shown that all paths $\phi_\sigma(t)$ intersect plane $z - x = 0$ for $t > 0$. Let $t_\sigma^{(2)} > 0$ be the first instant after $t = 0$ of the intersection of path $\phi_\sigma(t)$ with plane $z - x = 0$. From lemma 6.1 it follows that $0 < t_\sigma^{(1)} < t_\sigma^{(2)}$.

As was shown for the proof of lemma 3.3, path $\phi_\sigma(t)$ for $t > t_\sigma^{(2)}$ and sufficiently close to $t_\sigma^{(2)}$ lies in domain $\{x > 0, y > 0, z - x < 0\}$. Let $T'_\sigma \epsilon (t_\sigma^{(2)}, T_\sigma)$ be a number such that path $\phi_\sigma(t)$ lies in domain $\{y \geq 0, z \geq 0\}$ for $t \epsilon [t_\sigma^{(2)}, T'_\sigma]$. Then the following lemma is true.

Lemma 6.2

Suppose that inequalities $a > 0$, $b < 1$, $c^2 + b > 0$ and conditions (6.11)–(6.13) are fulfilled; then for $t \epsilon [t_\sigma^{(2)}, T'_\sigma]$, path $\phi_\sigma(t)$ lies in domain $\{x > \epsilon, y \geq 0, z - x < 0, z \geq 0\}$.

Proof

Evaluate $y_1(t_1^{(2)})$ as above. On interval $(0, t_1^{(2)})$, $z_1(t)$ is a decreasing function of time; therefore $z_1(t_1^{(2)}) < ck$, but $z_1(t_1^{(2)}) = x_1(t_1^{(2)})$ by definition of instant $t_1^{(2)}$; consequently,

$$x(t_1^{(2)}) < ck. \tag{6.26}$$

In consequence of equation (3.7), for $t \epsilon (0, t_1^{(2)})$ on path $\phi_1(t)$ there is fulfilled the inequality

$$\frac{dy}{dx} < \frac{ck}{y - f(x)}. \tag{6.27}$$

Now we will show that

$$y_1(t_1^{(2)}) < k + 2c^2k + 0.1. \tag{6.28}$$

Indeed, if for $t \epsilon [0, t_1^{(2)}]$ there is fulfilled the inequality

$$y_1(t) \leq k + c^2k + 0.1, \tag{6.29}$$

then inequality (6.28) follows immediately from inequality (6.29). Since $y_1(0) = k$ by definition of $y_1(t)$, then inequality (6.29) is fulfilled for $t = 0$. If this inequality is not fulfilled for all $t \epsilon [0, t_1^{(2)}]$, there exists a $t' \epsilon (0, t_1^{(2)})$ such that

$$y_1(t') = k + c^2k + 0.1, \tag{6.30}$$

and inequality (6.29) is fulfilled for $0 \leq t \leq t'$.

On segment $[0, t_1^{(2)}]$, $y_1(t)$ increases with time; therefore, for $t\epsilon(t', t_1^{(2)})$ it results that

$$y_1(t) > k + c^2k + 0.1. \tag{6.31}$$

From conditions (6.11)–(6.13) and from inequality (6.7), it follows that for $\alpha(x) < 0.1$, $0 \leq x \leq ck$. Because $x_1(t_1^{(2)}) < ck$ and $x_1(t)$ increases on interval $0 \leq t \leq t_1^{(2)}$, on $\phi_1(t)$ for $t\epsilon(0, t_1^{(2)})$ it results that $\alpha(x) < 0.1$. Therefore, in consequence of (6.27) and (6.31), on path $\phi_1(t)$ for $t\epsilon[t', t_1^{(2)}]$ there is fulfilled the inequality

$$\frac{dy}{dx} < \frac{ck}{k + c^2k + 0.1 - cx - 0.1}.$$

On $\phi_1(t)$ for $t\epsilon[0, t_1^{(2)}]$, $x < ck$; thus the last inequality produces

$$\frac{dy}{dx} < c.$$

Integrating this inequality along path $\phi_1(t)$ on interval $t' \leq t \leq t_1^{(2)}$, we obtain

$$y_1(t_1^{(2)}) - y_1(t') < cx_1(t_1^{(2)}).$$

Since $x_1(t_1^{(2)}) < ck$, the last inequality together with (6.30) gives (6.28).

If inequality (6.28) is used again, $y_1(t_1^{(2)})$ can be evaluated with considerably more precision. We will now find this more precise value of $y_1(t_1^{(2)})$. Dividing equation (5.39) by the first equation of system (2.15),

$$\frac{dw}{dx} = (1 - b)\frac{z - x}{y - f(x)}\alpha(x). \tag{6.32}$$

On path $\phi_1(t)$ for $t\epsilon[0, t_1^{(1)}]$ we have

$$\frac{dw}{dx} < \frac{(1 - b)\,ck}{k - c\varepsilon - H\varepsilon}Hx.$$

This results because $x_1(t)$ and $y_1(t)$ increase and $z_1(t)$ decreases with increasing time on the interval of time considered. Integrating the last inequality along path $\phi_1(t)$ from $t = 0$ to $t = t_1^{(2)}$, we obtain

$$w_1(t_1^{(1)}) - w_1(0) < \frac{(1 - b)\,ck}{2(k - c\varepsilon - H\varepsilon)}H\varepsilon^2. \tag{6.33}$$

We will show that

$$(1 - b)ckH\epsilon^2 < 0.01. \tag{6.34}$$

For $b \geq 0$, this inequality results because $ck = \max\{c, 1\}$ and because $c\epsilon < 0.1$, $H\epsilon < 0.1$ and $\epsilon < 0.1$, as inequalities (6.7) and (6.22) prove. Let $b < 0$; in this case, from (6.3) we obtain

$$ck\epsilon(1 - b)H\epsilon < ck\epsilon(H + c)\epsilon,$$

Therefore, (6.34) follows from (6.7). From (6.33), (6.34) and (6.7), we obtain

$$w_1(t_1^{(1)}) - w_1(0) < \frac{0{,}01}{2 \cdot 0{,}9} < 0{,}0056. \tag{6.35}$$

Dividing equation (5.39) by the second equation of system (2.15) we can write

$$\frac{dw}{dy} = (1 - b)\alpha(x). \tag{6.36}$$

However, $x_1(t)$ increases on the interval of time $[t_1^{(1)}, t_1^{(2)}]$; accordingly, from the definition of the instant of time $t_1^{(1)}$ and from (6.26), there follows inequality $\epsilon \leq x_1(t) \leq ck$ for $t \epsilon [t_1^{(1)}, t_1^{(2)}]$. As a consequence of condition (6.13), inequality $\alpha(x) < \delta^4$ is fulfilled on $\phi_1(t)$ for $t \epsilon [t_1^{(1)}, t_1^{(2)}]$. And thus from (6.36) there follows inequality

$$\frac{dw}{dy} < (1 - b)\delta^4, \tag{6.37}$$

which is true for path $\phi_1(t)$ for $t \epsilon [t_1^{(1)}, t_1^{(2)}]$. Integrating this inequality along path $\phi_1(t)$ from $t = t_1^{(1)}$ to $t = t_1^{(2)}$, we obtain

$$w_1(t_1^{(2)}) < w_1(t_1^{(1)}) + (1 - b)(y_1(t_1^{(2)}) - y_1(t_1^{(1)}))\delta^4.$$

Since $y_1(t_1^{(1)}) > y_1(0) = k$, from here and from inequality (6.28) we find

$$w_1(t_1^{(2)}) < w_1(t_1^{(1)}) + (1 - b)(2c^2k + 0{,}1)\delta^4.$$

Evaluating the second term on the right side of this equality, by hypothesis we have $-b < c^2$; therefore, we can write

$$\delta^4(1 - b)(2c^2k + 0{,}1) < (\delta^2 + c^2\delta^2)(2c^2k\delta^2 + 0{,}1\delta^2).$$

Following from this inequality,

$$\delta^4(1 - b)(2c^2k + 0.1) < (0.0025 + 0{,}01)(0{,}02 + 0{,}0005).$$

Thus, we obtain

$$w_1(t_1^{(2)}) < w_1(t_1^{(1)}) + 0{,}0003.$$

And from (6.35) we see that

$$\dot{w}_1(t_1^{(2)}) < w_1(0) + \dot{0},006. \tag{6.38}$$

From the definition of w in equality (5.38), we have

$$\frac{1}{2} y_1^2(t_1^{(2)}) < \frac{1}{2} k^2(1+c^2) + 0,006$$

or

$$y_1(t_1^{(2)}) < \sqrt{k^2(1+c^2)+0,012}.$$

Therefore, from lemma 3.9 there follows the correctness of the inequality

$$y_\sigma(t_\sigma^{(2)}) < \sqrt{k^2(1+c^2)+0,012} \quad \text{for} \quad \sigma \in [0,1]. \tag{6.39}$$

We will now prove that inequality (6.40) is correct for $t \epsilon [0, T_\sigma]$L

$$x_\sigma(t) < \frac{1}{c}\sqrt{k^2(1+c^2)+0,012}. \tag{6.40}$$

Indeed, proceeding from lemma 3.3, path $\phi_\sigma(t)$ intersects surface $y - f(x) = 0$ for $t \geq t_\sigma^{(2)}$. Let $t_\sigma^{(3)}$ be the first instant after $t_\sigma^{(1)}$ of the intersection of path $\phi_\sigma(t)$ with surface $y - f(x) = 0$. (We assume that this is the first existence of an intersection, i.e., that from domain $\{y - f(x) > 0\}$, path $\phi_\sigma(t)$ for $t = t_\sigma^{(3)}$ goes into domain $\{y - f(x) < 0\}$.) As was shown for the proof of lemma 3.3, $y_\sigma(t)$ on time interval $t \epsilon [0, t_\sigma^{(3)}]$ exhibits only one maximum for $t = t_\sigma^{(2)}$. Consequently, for $t \epsilon [0, t_\sigma^{(3)}]$,

$$y_\sigma(t) \leqslant y_\sigma(t_\sigma^{(2)}).$$

And therefore, from inequality (6.39) it follows that for $t \epsilon [0, t_\sigma^{(3)}]$ there takes place the inequality

$$y_\sigma(t) < \sqrt{k^2(1+c^2)+0,012} \quad \text{for} \quad \sigma \in [0, 1]. \tag{6.41}$$

Accordingly, for $t = t_\sigma^{(3)}$ and for all $\sigma \epsilon [0, 1]$, there is fulfilled inequality

$$y_\sigma = f(x_\sigma) < \sqrt{k^2(1+c^2)+0,012}.$$

But for $x > 0$, $f(x) > cx$; consequently,

$$x_\sigma(t_\sigma^{(3)}) < \frac{1}{c}\sqrt{k^2(1+c^2)+0,012}.$$

From this inequality and from lemma 3.9 it is easy to conclude the correctness of inequality (6.40) for all $t \epsilon [0, T_\sigma]$.

Now we will go immediately to the proof of lemma 6.2. To prove it, we will suppose to the contrary that the assertion of the lemma is not fulfilled. Then, since for $t > t_\sigma^{(2)}$ and sufficiently close to $t_\sigma^{(2)}$, $\phi_\sigma(t)$

lies in domain $\{x > \epsilon, y > 0, z - x < 0, z > 0\}$, from the continuity there exists a $t' \epsilon [t_\sigma^{(2)}, T'_\sigma]$ such that $\phi_\sigma(t) \epsilon \{x > \epsilon, y \geq 0, z - x < 0, z \geq 0\}$ for $t \epsilon (t_\sigma^{(2)}, t'_\sigma)$, and there is realized at least one of the equalities

$$x_\sigma(t') = \epsilon \text{ or } z_\sigma(t') - x_\sigma(t') = 0. \tag{6.42}$$

In the first place, suppose that the first of equalities (6.42) is fulfilled. Since $x_\sigma(t_\sigma^{(2)}) > \epsilon$, as follows from lemma 6.1, then on interval $t_\sigma^{(2)} \leq t < t'$, path $\phi_\sigma(t)$ intersects surface $y - f(x) = 0$ and goes into domain $\{y - f(x) < 0\}$. Thus, it results that $t_\sigma^{(3)} \epsilon [t_\sigma^{(2)}, t')$. Since path $\phi_\sigma(t)$ lies in domain $\{z - x < 0\}$ for $t \epsilon (t_\sigma^{(2)}, t')$, in consequence of the reasoning of section 3, instant $t_\sigma^{(3)}$ is the unique instant of intersection of path $\phi_\sigma(t)$ with surface $y - f(x) = 0$ on interval $[t_\sigma^{(2)}, t')$. Therefore, for $t = t'$, there must be fulfilled the inequality

$$y_\sigma - f(x_\sigma) < 0. \tag{6.43}$$

Because $x_\sigma(t') = \epsilon$, from the last inequality we obtain

$$y_\sigma(t') \leq c\epsilon + \delta^4. \tag{6.44}$$

Moreover, from that definition of the instant of time t', we have

$$0 \geqslant z_\sigma(t') - x_\sigma(t') \geqslant -x_\sigma(t') = -\varepsilon. \tag{6.45}$$

Making use of inequalities (6.44) and (6.45), evaluate $w_\sigma(t')$ from the above; since $y_\sigma(t') \geq 0$, we then will have

$$\begin{aligned} w_\sigma(t') &= \frac{1}{2} y_\sigma^2(t') + \frac{1}{2}(z_\sigma(t') - x_\sigma(t'))^2 \\ &< \frac{1}{2}(c\varepsilon + \delta^4)^2 + \frac{1}{2}\varepsilon^2. \end{aligned}$$

Thus, from inequalities (6.7) and (6.22) we see that

$$w_\sigma(t') < 0.03. \tag{6.46}$$

Following from (6.12) and (6.40), inequality $a(x) < \delta^4$ is fulfilled for $t \epsilon [t_\sigma^{(2)}, t']$ on $\phi_\sigma(t)$. And arising from (6.36) inequality (6.37) is fulfilled for $t \epsilon [t_\sigma^{(2)}, t']$ on $\phi_\sigma(t)$. Integrating this inequality along $\phi_\sigma(t)$ from $t = t_\sigma^{(2)}$ to $t = t'$, we obtain

$$w_\sigma(t') - w_\sigma(t_\sigma^{(2)}) > (1 - b)\delta^4 (y_\sigma(t') - y_\sigma(t_\sigma^{(2)})).$$

Since $y_\sigma(t') \geq 0$ from the definition of t', it results that

$$w_\sigma(t_\sigma^{(2)}) - w_\sigma(t') < (1 - b)\delta^4 (k + 2c^2k + 0.1).$$

And, as for the proof of inequality (6.38), we obtain

$$w_\sigma(t_\sigma^{(2)}) - w_\sigma(t') < 0.0003. \tag{6.47}$$

Owing to equality (5.39), $w_\sigma(t)$ increases on interval $[0, t_\sigma^{(2)}]$; consequently

$$w_\sigma(t_\sigma^{(2)}) > w_\sigma(0) \geqslant \frac{1}{2}.$$

Thus, from inequality (6.47) we have

$$w_\sigma(t') > \frac{1}{2} - 0{,}0003. \tag{6.48}$$

The last inequality contradicts inequality (6.46). The contradiction obtained proves that the first equality of (6.42) cannot be realized.

Suppose now that the second equality of (6.42) is realized. Following from the reasoning of section 3, path $\phi_\sigma(t)$ can intersect plane $z - x = 0$ for $t = t'$ only for the conditions that $y < f(x)$ and either relation (3.2) or (3.3) is fulfilled. We rewrite relations (3.2) and (3.3) in the following way:

$$\frac{-cx - ba(x)}{y - cx - a(x)} \leqslant 1$$

on path $\phi_\sigma(t)$ for $t = t'$. Since it must occur that $y < f(x)$ for $t = t'$ on $\phi_\sigma(t)$, from the last inequality it follows that

$$y - cx - a(x) \leq - cx - ba(x).$$

Thus we obtain

$$y \leq (1 - b)a(x). \tag{6.49}$$

But from the definition of the instant of time t', $x_\sigma(t') \geq \epsilon$; moreover, as mentioned already, inequality (6.40) is fulfilled for $t \epsilon [0, T_\sigma]$; therefore from (6.12) we see that

$$y_\sigma(t') \leq (1 - b)\delta^4.$$

Thus, as above, we obtain

$$y_\sigma(t') \leq 0.0003.$$

From this inequality and from the second equality in (6.42), we have

$$w_\sigma(t') \leqslant \frac{1}{2}(0.0003)^2. \tag{6.50}$$

On the other hand, in the case considered, inequality (6.48) is obviously true. Inequalities (6.48) and (6.50) are contradictory, and the contradictions obtained thus prove the lemma.

Now we explain in more detail the character of the behavior of trajectories $\phi_\sigma(t)$ for $\sigma \epsilon [0, 1]$. For $t \epsilon (0, t_\sigma^{(2)})$, path $\phi_\sigma(t)$, as follows from the definition of the instant of time $t_\sigma^{(2)}$, lies in domain $\{x > 0,\ z - x > 0,$

$y - f(x) > 0\}$; for $t = t_\sigma^{(2)}$, path $\phi_\sigma(t)$ intersects plane $z - x = 0$. For this, as results from lemma 6.1, $x_\sigma(t_\sigma^{(2)}) > \epsilon$. Further, path $\phi_\sigma(t)$ for $t_\sigma^{(3)} \geq t_\sigma^{(2)}$ intersects surface $y - f(x) = 0$ and goes into domain $\{x > 0,\ y - f(x) < 0,\ z - x < 0\}$. For an additional increase in t, $\phi_\sigma(t)$ cannot go to origin while remaining in domain $\{z - x < 0,\ y > 0,\ z > 0\}$ since, by lemma 6.2, inequality $x > \epsilon$ is fulfilled in this domain on $\phi_\sigma(t)$. Therefore, due to theorem 3.1, path $\phi_\sigma(t)$ for increasing time must leave domain $\{z - x < 0,\ y > 0,\ z > 0\}$. According to lemma 6.2, *path $\phi_\sigma(t)$ can leave this domain only through plane* $y = 0$ or plane $z = 0$.

Suppose first that it intersects plane $z = 0$ for $t > t_\sigma^{(2)}$. Then, according to lemma 3.1, path $\phi_\sigma(t)$ intersects plane $x = 0$ for $t = T_\sigma > t_\sigma^{(2)}$. On the path in this case, we find that $y_\sigma(T_\sigma) < 0$, $z_\sigma(T_\sigma) < 0$. Moreover, it is clear that path $\phi_\sigma(t)$ intersects plane $z - x = 0$ in this case only one time in interval $[0, T_\sigma]$ for $t = t_\sigma^{(2)}$; *it also intersects surface $y - f(x) = 0$ only one time for $t = t_\sigma^{(3)}$ (if a real intersection means a crossing of the* path from either domain $\{y - f(x) > 0\}$ or $\{y - f(x) < 0\}$ into the other). Plane $y = 0$ is intersected for $t\epsilon(0, T_\sigma)$ by path $\phi_\sigma(t)$ also only one time, for $t = t_\sigma^{(4)} > t_\sigma^{(3)}$. Since $\phi_\sigma(t)$ intersects surface $y - f(x) = 0$ only one time for $t = t_\sigma^{(3)}$ and since $x_\sigma(t_\sigma^{(3)}) \geq x_\sigma(t_\sigma^{(2)}) > \epsilon$, then plane $x = \epsilon$ is intersected by path $\phi_\sigma(t)$ only two times, for $t = t_\sigma^{(1)}$ and for $t = t_\sigma^{(5)} \epsilon (t_\sigma^{(3)}, T_\sigma)$.

Consider now the second case. Let path $\phi_\sigma(t)$ leave domain $\{x > 0,\ z - x < 0,\ y > 0,\ z > 0\}$ through plane $y = 0$. As before, let $t_\sigma^{(3)}$ be the first instant after $t_\sigma^{(1)}$ of the intersection of $\phi_\sigma(t)$ with surface $y - f(x) = 0$, and let $t_\sigma^{(4)}$ be the first instant after $t_\sigma^{(3)}$ of the intersection of $\phi_\sigma(t)$ with plane $y = 0$. If path $\phi_\sigma(t)$ does not intersect plane $z - x = 0$ for additional increases of t from $t_\sigma^{(4)}$ to T_σ, then, according to lemma 3.5 and the fact that $y_\sigma(t)$ then decreases for $t\epsilon[t_\sigma^{(4)}, T_\sigma]$, path $\phi_\sigma(t)$ intersects plane $z = 0$ for $t\epsilon(t_\sigma^{(4)}, T_\sigma]$ and furthermore, as follows from lemma 3.1, it intersects plane $x = 0$ for $t = T_\sigma$. For this, it results that $z_\sigma(T_\sigma) \leq 0$, $y_\sigma(T_\sigma) < 0$. In this case, as in the preceding, path $\phi_\sigma(t)$ for $t\epsilon(0, T_\sigma)$ intersects plane $z - x = 0$, surface $y - f(x) = 0$ and plane $y = 0$ only one time and plane $x = \epsilon$ two times, for $t = t_\sigma^{(1)}$ and for $t = t_\sigma^{(5)} \epsilon (t_\sigma^{(5)}, T_\sigma)$.

Consider now the last possibility. Let path $\phi_\sigma(t)$ intersect plane $z - x = 0$ for $t = t_\sigma^{(6)} \epsilon (t_\sigma^{(4)}, T_\sigma)$. For this we will say that $t_\sigma^{(6)}$ is the first instant after $t = t_\sigma^{(4)}$ of the intersection of $\phi_\sigma(t)$ with plane $z - x = 0$.

The following lemma is true.

Lemma 6.3

Let inequalities $a > 0$, $b < 1$, $c^2 + b > 0$ and conditions (6.11)–(6.13) be fulfilled; then inequality $y_\sigma(t) < 0$ is fulfilled for $t\epsilon[t_\sigma^{(6)}, T_\sigma]$.

Proof

Suppose first that $x_\sigma(t_\sigma^{(6)}) \geq \epsilon$. It is easy to see that on interval $t_\sigma^{(2)} \leq t \leq t_\sigma^{(6)}$, path $\phi_\sigma(t)$ intersects surface $y = f(x)$ only one time for $t = t_\sigma^{(3)}$; consequently, function $x_\sigma(t)$ on interval $[t_\sigma^{(2)}, t_\sigma^{(6)}]$ has only one maximum, at point $t_\sigma^{(3)}$. Since $x_\sigma(t_\sigma^{(2)}) > \epsilon$ and $x_\sigma(t_\sigma^{(6)}) \geq \epsilon$, then it results that $x_\sigma(t) \geq \epsilon$ for $t\epsilon[t_\sigma^{(2)}, t_\sigma^{(6)}]$. Moreover, inequality (6.40) is obviously true for $t\epsilon[t_\sigma^{(2)}, t_\sigma^{(6)}]$. In consequence of condition (6.12), on $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(2)}, t_\sigma^{(6)}]$ there is fulfilled inequality

$$\alpha(x) < \delta^4. \tag{6.51}$$

We evaluate $z_\sigma(t_\sigma^{(4)})$ below. Since by hypothesis we have $z_\sigma(t) - x_\sigma(t) < 0$ for $t \epsilon (t_\sigma^{(2)}, t_\sigma^{(6)})$, then on path $\phi_\sigma(t)$ for $t \epsilon (t_\sigma^{(2)}, t_\sigma^{(6)})$ it is true that

$$\frac{dz}{dy} > \frac{cx + b\alpha(x)}{x}$$

or

$$\frac{dz}{dy} > c + b\frac{\alpha(x)}{x}.$$

Because inequalities $x \geq \epsilon$ and (6.51) are satisfied on $\phi_\sigma(t)$ for $t \epsilon (t_\sigma^{(2)}, t_\sigma^{(6)})$, from the last inequality we conclude that

$$\frac{dz}{dy} > c + b\delta^3.$$

However, from the hypothesis of the lemma, $b > -c^2$; therefore, we can write

$$\frac{dz}{dy} > c(1 - c\delta^3).$$

Therefore, from (6.7) we obtain the following inequality true on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(2)}, t_\sigma^{(6)}]$:

$$\frac{dz}{dy} > 0{,}9\, c.$$

Integrating this inequality along path $\phi_\sigma(t)$ from $t_\sigma^{(2)}$ to $t_\sigma^{(4)}$, we obtain

$$z_\sigma(t_\sigma^{(4)}) - z_\sigma(t_\sigma^{(2)}) < 0{,}9\, c\,(y_\sigma(t_\sigma^{(4)}) - y_\sigma(t_\sigma^{(2)})).$$

By definition $y_\sigma(t_\sigma^{(4)}) = 0$; therefore, from the last inequality we come to

$$z_\sigma(t_\sigma^{(4)}) - z_\sigma(t_\sigma^{(2)}) < -0{,}9\, c y_\sigma(t_\sigma^{(2)}). \tag{6.52}$$

However, according to equality (5.39), function $w_\sigma(t)$ increases for $t \epsilon [0, t_\sigma^{(2)}]$; therefore we can write

$$\frac{1}{2} z_\sigma^2(0) \leqslant w_\sigma(0) < w_\sigma(t_\sigma^{(2)}) = \frac{1}{2} y_\sigma^2(t_\sigma^{(2)}).$$

Since $z_\sigma(0) = ck$, we obtain

$$y_\sigma(t_\sigma^{(2)}) > ck. \tag{6.53}$$

And from (6.52) we come to

$$z_\sigma(t_\sigma^{(4)}) - z_\sigma(t_\sigma^{(2)}) < -0{,}9\, c^2 k.$$

But for $t \in [0, T_\sigma]$, function $z_\sigma(t)$ is a decreasing function of time; consequently,

$$z_\sigma(t_\sigma^{(4)}) < ck - 0{,}9\,c^2k = ck\,(1 - 0{,}9\,c). \tag{6.54}$$

Now evaluate $w_\sigma(t_\sigma^{(6)})$. On the interval of time $t_\sigma^{(2)} \leq t \leq t_\sigma^{(6)}$ on $\phi_\sigma(t)$, inequality (6.51) is true and with it (6.37), also. By integrating this inequality along $\phi_\sigma(t)$ on interval $t_\sigma^{(2)} \leq t \leq t_\sigma^{(6)}$, we obtain

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > (1 - b)\,\delta^4\,(y_\sigma(t_\sigma^{(6)}) - y_\sigma(t_\sigma^{(2)})).$$

Since $w_\sigma(t)$ decreases with increasing time for $t \in [t_\sigma^{(2)}, t_\sigma^{(6)}]$, then $|y_\sigma(t_\sigma^{(6)})| < y_\sigma(t_\sigma^{(2)})$. Thus from (6.39) we have

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > -2\delta^4(1 - b)\sqrt{k^2(1 + c^2) + 0{,}012}.$$

Because $k^2(1 + c^2) > 1$ and $w_\sigma(t_\sigma^{(2)}) > w_\sigma(0) \geq \frac{1}{2}c^2k^2$, from the last inequality we obtain

$$w_\sigma(t_\sigma^{(6)}) > \frac{1}{2}c^2k^2 - 3k^2(1 + c^2)(1 - b)\,\delta^4.$$

Therefore, making use of inequality (6.7), it is easy to obtain the estimate

$$w_\sigma(t_\sigma^{(6)}) > \frac{1}{2}c^2k^2(1 - 0{,}06\,\delta^2). \tag{6.55}$$

Suppose now, contrary to the assertion of the lemma, that there exists a $\tau_\sigma \in (t_\sigma^{(6)}, T_\sigma]$ such that $y_\sigma(\tau_\sigma) = 0$ and for $t \in [t_\sigma^{(6)}, \tau_\sigma)$

$$y_\sigma(t) < 0. \tag{6.56}$$

Then, according to the reasoning of section 3, $z_\sigma(t) - x_\sigma(t) > 0$ for $t \in (t_\sigma^{(6)}, \tau_\sigma)$. Thus, from (5.39) it follows that

$$w_\sigma(\tau_\sigma) > w_\sigma(t_\sigma^{(6)}). \tag{6.57}$$

Since $z_\sigma(t)$ decreases for $t \in [0, T_\sigma]$

$$z_\sigma(\tau_\sigma) < z_\sigma(t_\sigma^{(4)}). \tag{6.58}$$

From inequalities (6.55) and (6.57) we see that

$$w_\sigma(\tau_\sigma) > \frac{1}{2}c^2k^2(1 - 0{,}06\delta^2). \tag{6.59}$$

On the other hand, since $y_\sigma(\tau_\sigma) = 0$, then from (6.58) we find

$$w_\sigma(\tau_\sigma) < \frac{1}{2}z_\sigma^2(t_\sigma^{(4)}),$$

and thus from (6.54) we obtain

$$w_\sigma(\tau_\sigma) < \frac{1}{2} c^2k^2(1-0.9c)^2. \tag{6.60}$$

We will show that inequalities (6.59) and (6.60) are contradictory. To do this we will attempt to prove that

$$(1 - 0.9c)^2 < 1 - 0.06\delta^2. \tag{6.61}$$

We will show this. From the conditions of the case, $z_\sigma(t_\sigma^{(4)}) > 0$; therefore, $1 - 0.9c > 0$ because of (6.54). Accordingly, from inequality (6.7), (6.61) also follows. Thus, inequalities (6.59) and (6.60) are contradictory. This contradiction proves the lemma in the case $x_\sigma(t_\sigma^{(6)}) \geq \epsilon$.

Suppose now that $x_\sigma(t_\sigma^{(6)}) < \epsilon$. Since $x_\sigma(t_\sigma^{(3)}) > \epsilon$, there exists in this case a unique $t_\sigma^{(5)} \epsilon (t_\sigma^{(3)}, t_\sigma^{(6)})$ for which $x_\sigma(t_\sigma^{(5)}) = \epsilon$. For this it is obvious that there is fulfilled the inequality

$$z_\sigma(t_\sigma^{(5)}) - x_\sigma(t_\sigma^{(5)}) < 0. \tag{6.62}$$

Then as in the preceding case, it is easy to ascertain that on the interval of time $t_\sigma^{(2)} \leq t \leq t_\sigma^{(5)}$ there is fulfilled inequality (6.51) and, with it, (6.37) also. Integrating this inequality, we obtain inequality (6.55). From this inequality and from (6.7) there follows the relation

$$w_\sigma(t_\sigma^{(5)}) > \frac{1}{2} c^2k^2 - 0.0003. \tag{6.63}$$

By hypothesis, path $\phi_\sigma(t)$ intersects plane $z - x = 0$ for $t = t_\sigma^{(6)} \epsilon (t_\sigma^{(5)}, T_\sigma)$; consequently, $z_\sigma(t_\sigma^{(5)}) > 0$. Therefore, from (6.62), from the definition of $t_\sigma^{(5)}$ and from (6.63), we have

$$y_\sigma^2(t_\sigma^{(5)}) + \varepsilon^2 > c^2k^2 - 0.0006.$$

Thus it follows that

$$y_\sigma(t_\sigma^{(5)}) < -0.7. \tag{6.64}$$

Since $y_\sigma(t)$ decreases with increasing time for $t \epsilon [t_\sigma^{(5)}, t_\sigma^{(6)}]$, from the last inequality we obtain

$$y_\sigma(t_\sigma^{(6)}) < -0.7. \tag{6.65}$$

Suppose to the contrary of the lemma's assertion that there exists a $\tau_\sigma \epsilon (t_\sigma^{(6)}, T_\sigma]$ such that $y_\sigma(\tau_\sigma) = 0$, and inequality (6.56) is fulfilled for $t \epsilon (t_\sigma^{(6)}, \tau_\sigma)$. Then it results that $z_\sigma(t) - x_\sigma(t) > 0$ for $t \epsilon (t_\sigma^{(6)}, \tau_\sigma)$. Thus inequality (6.57) follows. It is not difficult to see that

$$0 \leqslant z_\sigma(\tau_\sigma) < z_\sigma(t_\sigma^{(5)}) \leqslant \varepsilon.$$

Consequently we obtain

$$w_\sigma(\tau_\sigma) < \frac{1}{2} \varepsilon^2. \tag{6.66}$$

From inequality (6.57) and from (6.55) it follows that

$$w_\sigma(\tau_\sigma) > \frac{1}{2}(0.7)^2. \tag{6.67}$$

This inequality contradicts inequality (6.66), and the contradiction obtained proves lemma 6.3.

According to lemma 6.3 any path $\phi_\sigma(t)$ intersects plane $x = 0$ for $t = T_\sigma$, where T_σ is a finite number. Indeed, if instant $t_\sigma^{(6)}$ is not defined on $\phi_\sigma(t)$ (i.e., if $\phi_\sigma(t)$ does not intersect plane $z - x = 0$ for $t \epsilon (t_\sigma^{(3)}, T_\sigma)$), then it was proved above that $\phi_\sigma(t)$ intersects plane $x = 0$ for $t = T_\sigma$, and for this it results that $y_\sigma(T_\sigma) < 0$. But if instant $t_\sigma^{(6)}$ is defined, then path $\phi_\sigma(t)$ lies in domain $\{x > 0,\ y < 0,\ z - x > 0\}$ for $t \epsilon (t_\sigma^{(6)}, T_\sigma)$. However, path $\phi_\sigma(t)$ cannot remain in this domain for all $t > t_\sigma^{(6)}$ since by theorem 3.1 it would then have to go to origin, but this is impossible for function $w_\sigma(t)$ always increases with time in domain $\{x > 0,\ z - x > 0\}$. Consequently, path $\phi_\sigma(t)$ goes into domain $\{x > 0,\ y < 0,\ z - x > 0\}$. This path cannot intersect plane $y = 0$; therefore, it also cannot intersect plane $z - x = 0$ for $t \epsilon [t_\sigma^{(6)}, T_\sigma]$, since for $y < 0$ and $x \geq 0$, as seen from the reasoning of section 3, all paths of system (2.15) intersect plane $z - x = 0$, crossing from domain $\{z - x < 0\}$ into domain $\{z - x > 0\}$, and not the reverse way. Consequently, path $\phi_\sigma(t)$ intersects plane $x = 0$ for a finite T_σ, and for this according to lemma 6.3 it results that $y_\sigma(T_\sigma) < 0$.

Thus, path $\phi_\sigma(t)$ behaves in the following way. For increasing time from $t = 0$ to $t = t_\sigma^{(2)}$, $\phi_\sigma(t)$ lies in domain $\{x > 0,\ y - f(x) > 0,\ z - x > 0\}$. For $t = t_\sigma^{(2)}$, path $\phi_\sigma(t)$ intersects plane $z - x = 0$ and crosses into domain $\{x > 0,\ z - x < 0\}$. Finally, for $t = t_\sigma^{(3)} \geq t_\sigma^{(2)}$, path $\phi_\sigma(t)$ intersects surface $y - f(x) = 0$ and goes into domain $\{x > 0,\ y - f(x) < 0,\ z - x < 0\}$. For a further increase of time, path $\phi_\sigma(t)$ intersects plane $y = 0$ for $t = t_\sigma^{(4)}$. Further, one of two things is possible: either $\phi_\sigma(t)$ intersects plane $z - x = 0$ for $t = t_\sigma^{(6)}$ and then plane $x = 0$ for $t = T_\sigma > t_\sigma^{(6)}$, or $\phi_\sigma(t)$ does not intersect plane $z - x = 0$, and does intersect plane $x = 0$ for $t = T_\sigma > t_\sigma^{(4)}$. For this, $y_\sigma(T_\sigma) < 0$ in both cases. Since function $x_\sigma(t)$ shows only one maximum on the interval of time $0 \leq t \leq T_\sigma$, path $\phi_\sigma(t)$ intersects the plane $x = \epsilon$ only two times on interval $0 < t < T_\sigma$ for $t = t_\sigma^{(1)}$ and for $t = t_\sigma^{(5)} > t_\sigma^{(1)}$.

In the following discourses we will, as before, suppose that inequalities $a > 0$, $b < 1$, $c^2 + b > 0$ and conditions (6.11)–(6.13) are fulfilled.

We will consider the function

$$u = c^2 x - cy + z. \tag{6.68}$$

Obviously, its time derivative is equal to

$$\dot{u} = -cu - (c^2 + b)\alpha(x). \tag{6.69}$$

Below, we estimate the quantity $u_\sigma(t_\sigma^{(5)})$. From equality (6.69) for $u \leq 0$, we obtain

$$\frac{du}{dt} \geqslant -(c^2 + b)\alpha(x), \tag{6.70}$$

Dividing this inequality by the equation of (2.15), for $t \epsilon [0, t_\sigma^{(1)}]$ and $u \leq 0$ on path $\phi_\sigma(t)$, we will have

$$\frac{du}{dt} \geqslant -\frac{(c^2+b)\,\alpha(x)}{y-f(x)}. \tag{6.71}$$

But for $u \leq 0$, $y \geq cx + \frac{1}{c}z$. Therefore, inequality (6.71) gives

$$\frac{du}{dx} \geqslant -\frac{(c^2+b)\,\alpha(x)}{\frac{z}{c}-\alpha(x)} \geqslant -\frac{(c^2+b)\,Hx}{\frac{z}{c}-H\varepsilon}$$

on path $\phi_\sigma(t)$ for $t\epsilon[0, t_\sigma^{(1)}]$ and for $u \leq 0$. However, estimate (6.25) occurs on the interval of time $0 \leq t \leq t_\sigma^{(1)}$; therefore, from the last inequality and from inequality (6.7) we have

$$\frac{du}{dx} > -\frac{(c^2+b)\,Hx}{0.8}.$$

From the definition of point p_σ, $u_\sigma(0) \geq 0$; having noted this, we integrate the last inequality along path $\phi_\sigma(t)$ on interval $[0, t_\sigma^{(1)}]$ and obtain

$$u_\sigma(t_\sigma^{(1)}) - u_\sigma(0) > -\frac{H\varepsilon^2}{1.6}(c^2+b).$$

Since $u_\sigma(0) \geq 0$, then

$$u_\sigma(t_\sigma^{(1)}) > -\frac{H\varepsilon^2}{1.6}(c^2+b). \tag{6.72}$$

Dividing inequality (6.70) by the third equation of system (4.7) we find that

$$\frac{du}{dz} \leqslant \frac{(c^2+b)\,\alpha(x)}{cx+b\alpha(x)} \tag{6.73}$$

for $u \leq 0$. For $t\epsilon[t_\sigma^{(1)}, t_\sigma^{(5)}]$ we have inequalities $x_\sigma(t) \geq \epsilon$ and (6.40). Thus, because of (6.13),

$$\frac{du}{dz} < \frac{(c^2+b)\,\delta^4}{cx+b\alpha(x)}.$$

By supposition, $c^2 + b > 0$; therefore, from the last inequality we have

$$\frac{du}{dz} < \frac{(c^2+b)\,\delta^4}{cx-c^2\alpha(x)}.$$

Since the inequalities $x \geq \epsilon$ and $\alpha(x) < \delta^4$ takes place on $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(1)}, t_\sigma^{(5)}]$, we can write

$$\frac{du}{dz} < \frac{\left(c+\frac{b}{c}\right)\delta^3}{1-c\delta^3}. \tag{6.74}$$

On interval $t_\sigma^{(1)} \le t \le t_\sigma^{(5)}$, $z_\sigma(t)$ decreases with increasing time; accordingly, by integrating inequality (6.74) along $\phi_\sigma(t)$ on interval $[t_\sigma^{(1)}, t_\sigma^{(5)}]$, we obtain

$$u_\sigma(t_\sigma^{(5)}) - u_\sigma(t_\sigma^{(1)}) > \frac{c + \frac{b}{c}}{1 - c\delta^3}\delta^3 (z_\sigma(t_\sigma^{(5)}) - z_\sigma(t_\sigma^{(1)})). \tag{6.75}$$

Since $z_\sigma(t)$ decreases for $t \epsilon [0, T_\sigma]$,

$$z_\sigma(t_\sigma^{(1)}) < z_\sigma(0) = ck. \tag{6.76}$$

We will show that

$$z_\sigma(t_\sigma^{(5)}) > -\sqrt{k^2(1 + c^2) + 0.012}. \tag{6.77}$$

Indeed, if path $\phi_\sigma(t)$ intersects plane $z - x = 0$ for $t = t_\sigma^{(6)} \epsilon (t_\sigma^{(3)}, T_\sigma]$, then as proved above, $z_\sigma(t_\sigma^{(5)}) > 0$ and inequality (6.77) is obvious. But if path $\phi_\sigma(t)$ does not intersect plane $z - x = 0$ for $t \epsilon [t_\sigma^{(3)}, T_\sigma]$, then as proved earlier, $z_\sigma(t) - x_\sigma(t) \le 0$ for $t_\sigma^{(2)} \le t \le T_\sigma$. Thus according to (5.39), $w_\sigma(t)$ decreases with increasing /121
time for $t \epsilon [t_\sigma^{(2)}, T_\sigma]$; however, from inequality (6.39) and from the definition of function w, it follows that

$$w_\sigma(t_\sigma^{(5)}) < w_\sigma(t_\sigma^{(2)}) < \frac{1}{2} k^2 (1 + c^2) + 0.006.$$

But for $z_\sigma(t_\sigma^{(5)}) < 0$ we have $z_\sigma^2(t_\sigma^{(5)}) < 2w_\sigma(t_\sigma^{(5)})$. Therefore, from the preceding inequality we have (6.77). And from inequalities (6.75)–(6.77) we obtain

$$u_\sigma(t_\sigma^{(5)}) - u_\sigma(t_\sigma^{(1)}) > \frac{-\left(c + \frac{b}{c}\right)\delta^3}{1 - c\delta^3}\left[ck + \sqrt{k^2(1 + c^2) + 0.012}\right].$$

Thus, it is easy to verify the inequality

$$u_\sigma(t_\sigma^{(5)}) - u_\sigma(t_\sigma^{(1)}) > -\frac{c + \frac{b}{c}}{1 - c\delta^3}\delta^3 (2ck + k). \tag{6.78}$$

Since $b < 1$ by hypothesis, from this inequality and from inequality (6.22) it is easy to see the following inequality:

$$u_\sigma(t_\sigma^{(5)}) - u_\sigma(t_\sigma^{(1)}) > -0.0035c. \tag{6.79}$$

From inequality (6.72) because of (6.7) we have

$$u_\sigma(t_\sigma^{(1)}) > -0.0063c.$$

And thus from (6.79) we conclude

$$u_\sigma(t_\sigma^{(5)}) > -0.01c. \tag{6.80}$$

We will show that

$$w_\sigma(t_\sigma^{(5)}) > \frac{1}{2}k^2(\sigma^2 + c^2) - 0{,}0002c^2. \tag{6.81}$$

Suppose first that path $\phi_\sigma(t)$ intersects plane $z - x = 0$ for $t = t_\sigma^{(6)} \epsilon (t_\sigma^{(3)}, T_\sigma]$. Moreover, suppose that $t_\sigma^{(6)} \leq t_\sigma^{(5)}$. Then, as was proved earlier, $z_\sigma(t) - x_\sigma(t) > 0$ on the interval $t_\sigma^{(6)} \leq t \leq t_\sigma^{(5)}$; consequently, $w_\sigma(t_\sigma^{(6)}) < w_\sigma(t_\sigma^{(5)})$. Therefore, for the proof of inequality (6.81) we need to establish the inequality

$$w_\sigma(t_\sigma^{(6)}) > \frac{1}{2}k^2(\sigma^2 + c^2) - 0{,}0002c^2. \tag{6.82}$$

On the interval $[t_\sigma^{(2)}, t_\sigma^{(6)}]$, $w_\sigma(t)$ decreases; consequently,

$$y_\sigma(t_\sigma^{(6)}) > -\sqrt{k^2(1 + c^2) + 0{,}012}. \tag{6.83}$$

Furthermore, on interval $t_\sigma^{(2)} \leq t \leq t_\sigma^{(6)}$, because $t_\sigma^{(5)} \geq t_\sigma^{(6)}$, on path $\phi_\sigma(t)$ inequality (6.37) is fulfilled. Integrating this we obtain

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > (1 - b)\,\delta^4\,(y_\sigma(t_\sigma^{(6)}) - y_\sigma(t_\sigma^{(2)})).$$

Therefore, from (6.39) and (6.83) we obtain

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > -2(1 - b)\,\delta^4\sqrt{k^2(1 + c^2) + 0{,}012.}$$

And thus we come to

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > -2(1 - b)\,k\,(1 + c)\,\delta^4. \tag{6.84}$$

From this inequality with the help of inequality (6.7), it is not hard to obtain

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(2)}) > 0.0002c^2.$$

However, $w_\sigma(t)$ increases for $t\epsilon[0, t_\sigma^{(2)}]$; therefore,

$$w_\sigma(t_\sigma^{(2)}) > w_\sigma(0) = \frac{1}{2}k^2(\sigma^2 + c^2).$$

Thus, we also obtain (6.82) and with it (6.81) besides.

When $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(2)}, t_\sigma^{(5)}]$ does not intersect plane $z - x = 0$, the proof of inequality (6.81) proceeds analogously except that instant $t_\sigma^{(5)}$ should be considered instead of instant $t_\sigma^{(6)}$.

We introduce the following designations. First assume that

$$\eta^2 = k^2\frac{\sigma^2 + c^2}{1 + c^2} \quad \eta > 0. \tag{6.85}$$

We designate

$$y_\sigma(t_\sigma^{(5)}) = -\eta + \delta y_\sigma, \tag{6.86}$$

$$z_\sigma(t_\sigma^{(5)}) = -c\eta + \delta z_\sigma. \tag{6.87}$$

Then inequality (6.80) takes on the following form:

$$-c\delta y_\sigma + \delta z_\sigma > -c^2\epsilon - 0.01c. \tag{6.88}$$

But inequality (6.81) is then rewritten in the form

$$(-c\eta + \delta z_\sigma - \epsilon)^2 + (-\eta + \delta y_\sigma)^2 - (1 + c^2)\eta^2 > -0.0004c^2.$$

From this inequality we obtain

$$\begin{aligned} &-2c\eta\delta z_\sigma - 2\eta\delta y_\sigma + \delta z_\sigma^2 + \delta y_\sigma^2 - 2\varepsilon\delta z_\sigma \\ &> -2c\eta\varepsilon - \varepsilon^2 - 0.0004c^2. \end{aligned} \tag{6.89}$$

Evaluate $\frac{dy}{dx} = \frac{z-x}{y-f(x)}$ on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(5)}, T_\sigma]$. The following four cases are possible.

1. $\delta y_\sigma \geq 0$, $\delta z_\sigma \leq 0$.

In this case, from (6.88) we obtain

$$\delta y_\sigma < c\epsilon + 0.01 < 0.11, \tag{6.90}$$

$$\delta z_\sigma > -c^2\epsilon - 0.01c + c\delta y_\sigma. \tag{6.91}$$

Consequently, on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(5)}, T_\sigma]$ there is fulfilled the inequality

$$\frac{dz}{dx} = \frac{-cx - b\alpha(x)}{y - f(x)} \leqslant \frac{-cx - b\alpha(x)}{-\eta + \delta y_\sigma - f(x)} \leqslant \frac{-cx - b\alpha(x)}{-\eta + 0{,}11}.$$

From equality (6.85) it is easy to verify the relation $\eta \geq \frac{\sqrt{2}}{2} > 0.7$; therefore,

$$\frac{dz}{dx} < 2(cx + b\alpha(x)) < 2(cx + Hx),$$

since $\alpha(x) \leq Hx$ and $b < 1$ for $0 \leq x \leq \epsilon$. Integrating the last inequality along path $\phi_\sigma(t)$ from $t_\sigma^{(5)}$ to T_σ, we obtain

$$z_\sigma(T_\sigma) - z_\sigma(t_\sigma^{(5)}) > -(c\varepsilon^3 + H\varepsilon^2)$$

or

$$z_\sigma(T_\sigma) > -0.1\epsilon - c\eta + \delta z_\sigma. \tag{6.92}$$

Therefore, from (6.91) we obtain for path $\phi_\sigma(t)$ on the interval of time $[t_\sigma^{(s)}, T_\sigma]$,

$$\frac{z-x}{y-f(x)} < \frac{c\eta - c\delta y_\sigma + 0{,}1\varepsilon + c^2\varepsilon + 0{,}01c + \varepsilon}{\eta - \delta y_\sigma}.$$

Accordingly,

$$\frac{z-x}{y-f(x)} < c + \frac{0{,}1\varepsilon + c^2\varepsilon + 0{,}01c + \varepsilon}{\eta - \delta y_\sigma}.$$

Since $\eta \geq 0.7$, from (6.7) and (6.90) we obtain

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < 1.38c \tag{6.93}$$

on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$.

2. $\delta y_\sigma \leq 0$, $\delta z_\sigma \geq 0$.

In this case, on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$, the following inequality is obviously fulfilled:

$$\frac{dz}{dx} < \frac{-cx - b\alpha(x)}{-\eta + \delta y_\sigma - f(x)} < 2(cx + Hx).$$

Consequently,

$$z_\sigma(T_\sigma) > -c\eta - c\epsilon^2 - H\epsilon^2 + \delta z_\sigma.$$

Accordingly, in this case on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$, there is fulfilled the inequality

$$\frac{z-x}{y-f(x)} < \frac{-c\eta - \varepsilon - \varepsilon^2(c+H)}{-\eta}$$

or

$$\frac{z-x}{y-f(x)} < c + c\frac{0{,}11}{\eta}.$$

Thus, following from $\eta > 0.7$, inequality (6.93) is also fulfilled in case 2.

3. $\delta y_\sigma \geq 0$, $\delta z_\sigma \geq 0$.

As was proved above, $y < f(x)$ on path $\phi_\sigma(t)$ for $t = t_\sigma^{(5)}$, but $x_\sigma(t_\sigma^{(5)}) = \epsilon$ by definition of the instant of time $t_\sigma^{(5)}$; therefore,

$$y_\sigma(t_\sigma^{(5)}) < c\epsilon + H\epsilon < 0.1.$$

Thus, from (6.90) there results inequality

$$\delta y_\sigma < \eta + 0.1.$$

Since $\eta > 0.7$, from the last inequality it follows that

$$\delta y_\sigma < 1.5\eta.$$

And consequently $2\eta\delta y_\sigma > \delta y_\sigma^2$. Proceeding from this inequality and from inequality (6.89),

$$\delta y_\sigma^2 - 2(c\eta + \epsilon)\delta z_\sigma + 2c\eta\epsilon + \epsilon^2 + 0.0004c^2 > 0.$$

In consequence of this quadratic inequality for δz_σ, there is fulfilled one of two inequalities:

$$\delta z_\sigma > c\eta + \varepsilon + \sqrt{c^2\eta^2 - 0{,}0004c^2} \tag{6.94}$$

or

$$\delta z_\sigma < c\eta + \varepsilon - \sqrt{c^2\eta^2 - 0{,}0004c^2}. \tag{6.95}$$

Go first to the case when inequality (6.94) is fulfilled. From (6.94), because $\eta > 0.7$, we obtain

$$\delta z_\sigma > c\eta + \epsilon + 0.996c\eta > 1.996c\eta.$$

Thus, from the definition of δz_σ it follows that

$$z_\sigma(t_\sigma^{(5)}) > 0.996c\eta. \tag{6.96}$$

But from equality (6.85) we have

$$\eta \geqslant \frac{kc}{\sqrt{1+c^2}}. \tag{6.97}$$

Therefore, it is not difficult to verify the following inequality:

$$0.996c\eta > \epsilon. \tag{6.98}$$

We will show that

$$0{,}9\,\frac{kc^2}{\sqrt{1+c^2}} > \varepsilon. \tag{6.99}$$

For $c \geq 1$, inequality (6.99) results because $k = \max\left\{1, \frac{1}{c}\right\} = 1$ and because $\frac{c^2}{\sqrt{1+c^2}} > \frac{1}{2}$. For $c < 1$, inequality (6.99) follows since $k = \max\left\{1, \frac{1}{c}\right\} = \frac{1}{c}$ and since $\sqrt{\varepsilon^2\left(1+\frac{1}{c^2}\right)} < 0.2$ by virtue of (6.7) and (6.22). From inequality (6.99) we obtain inequality (6.98). The following inequality results from inequalities (6.96) and (6.98):

$$z_\sigma(t_\sigma^{(s)}) - \epsilon > 0.$$

But from the definition of $t_\sigma^{(s)}$ we have $x_\sigma(t_\sigma^{(s)}) = \epsilon$; consequently,

$$z_\sigma(t_\sigma^{(s)}) - x_\sigma(t_\sigma^{(s)}) > 0. \tag{6.100}$$

As proven earlier, this inequality is possible only when for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$ there are fulfilled the inequalities

$$z_\sigma(t) - x_\sigma(t) > 0,\ y_\sigma(t) < 0. \tag{6.101}$$

From these inequalities we see that on $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$ there is fulfilled the inequality

$$\frac{z-x}{y-f(x)} < 0. \tag{6.102}$$

Now consider the case when inequality (6.95) is fulfilled. From this inequality by virtue of $\eta > 0.7$ there follows inequality

$$\delta z_\sigma < c\eta + \epsilon - c\eta + 0.0004c \tag{6.103}$$

or

$$\delta y_\sigma < 0.1004c. \tag{6.104}$$

From inequality (6.88) we obtain

$$c\delta y_\sigma < \delta z_\sigma + c^2\epsilon + 0.01c. \tag{6.105}$$

Arising from (6.104) we have

$$\delta y_\sigma < 0.211. \tag{6.106}$$

Accordingly, on $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$, there occurs the inequality $\frac{dz}{dx} = \frac{-cx - b\alpha(x)}{y - f(x)} < \frac{cx + Hx}{\eta - 0{,}211}$;

since $\eta > 0.7$, then

$$\frac{dz}{dx} < \frac{cx + Hx}{0{,}4\delta9}.$$

Integrating the last inequality along path $\phi_\sigma(t)$ from $t = t_\sigma^{(s)}$ to $t = T_\sigma$, we obtain

$$z_\sigma(T_\sigma) > -c\eta - \frac{1}{0{,}978}(c\varepsilon^2 + H\varepsilon^2) + \delta z_\sigma.$$

Proceeding from this inequality, on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$ the following inequality is fulfilled:

$$\frac{z - x}{y - f(x)} < \frac{c\eta - \delta z_\sigma + \varepsilon + 0{,}11\varepsilon}{\eta - \delta y_\sigma}.$$

Thus, from (6.105) there results the inequality

$$\frac{z - x}{y - f(x)} < c\,\frac{\eta - \dfrac{\delta z_\sigma}{c} - c\varepsilon - 0{.}01 + c\varepsilon + 0{,}01 + \dfrac{\varepsilon}{c} + 0{,}11\dfrac{\varepsilon}{c}}{\eta - \dfrac{\delta z_\sigma}{c} - c\varepsilon - 0{,}01}.$$

Accordingly, from (6.7) and (6.22) we obtain

$$\frac{z - x}{y - f(x)} < c + \frac{0{,}121}{\eta - \dfrac{\delta z_\sigma}{c} - c\varepsilon - 0{,}01}\,c.$$

And thus, because of (6.105) we conclude that on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$ the following inequality is true:

$$\frac{z - x}{y - f(x)} < c + \frac{0{,}121}{0{,}489}\,c < 1{,}25c.$$

Consequently, inequality (6.93) is also fulfilled in the case considered.

4. $\delta y_\sigma \leq 0$, $\delta z_\sigma \leq 0$.

This case we will not consider.

Thus, inequality (6.93) is true in cases 1, 2, and 3. In consequence of this inequality, equality (6.32) and the conditions of (6.11) and (6.12), it follows that inequality

$$\frac{dw}{dx} < 1{,}38\,(1 - b)\,cHx, \tag{6.107}$$

is true for path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(5)}, T_\sigma]$. Integrating this inequality along $\phi_\sigma(t)$ on this interval, we obtain

$$w_\sigma(T_\sigma) - w_\sigma(t_\sigma^{(5)}) > -0.69(1-b)Hc\epsilon^2. \tag{6.108}$$

Turn to inequality (6.84). This inequality because of condition (6.7) can be put in the form

$$w_\sigma(t_\sigma^{(6)}) - w_\sigma(t_\sigma^{(3)}) > -0.4(1-b)\delta^3. \tag{6.109}$$

But $w_\sigma(t)$ increases with increasing time for $t \epsilon [t_\sigma^{(6)}, T_\sigma]$; therefore, if $t_\sigma^{(5)} \geq t_\sigma^{(6)}$, from (6.109) we obtain

$$w_\sigma(t_\sigma^{(5)}) > w_\sigma(t_\sigma^{(2)}) - 0.4(1-b)\delta^3. \tag{6.110}$$

However, if $t_\sigma^{(5)} < t_\sigma^{(6)}$ or if $t_\sigma^{(6)}$ is not defined on $\phi_\sigma(t)$, inequality (6.110) is obtained in the same way as inequality (6.84), except that instant $t_\sigma^{(5)}$ ought to be considered instead of $t_\sigma^{(6)}$. Since $w_\sigma(t_\sigma^{(2)}) > w_\sigma(0) = \frac{1}{2}(1 + c^2)\eta^2$, from inequality (6.110) (because $cH > 1$) we obtain

$$w_\sigma(t_\sigma^{(5)}) > \frac{1}{2}(1 + c^2)\eta^2 - 0.02(1-b)cH\delta^2.$$

From the last inequality and inequality (6.108) we have

$$w_\sigma(T_\sigma) > \frac{1}{2}(1 + c^2)\eta^2 - 0.71(1-b)cH\epsilon^2. \tag{6.111}$$

From inequality (6.78), with the assistance of (6.7), (6.22) and (6.2) or (6.3), it is easy to obtain the inequality

$$u_\sigma(t_\sigma^{(5)}) - u_\sigma(t_\sigma^{(1)}) > -0.3H(c^2 + b)\epsilon^2. \tag{6.112}$$

And from this inequality and from (6.72),

$$u_\sigma(t_\sigma^{(5)}) > -0.7H(c^2 + b)\epsilon^2. \tag{6.113}$$

Because of inequality (6.70), on $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(5)}, T_\sigma]$ and for $u \leq 0$ there is fulfilled the inequality

$$\frac{du}{dx} < -\frac{c^2 + b}{y - f(x)} Hx.$$

However, inequality (6.106) takes place in cases 1, 2, and 3. Therefore, from the last inequality we obtain

$$\frac{du}{dx} < -\frac{(c^2 + b)\, Hx}{\eta - 0{,}211} < -\frac{(c^2 + b)\, Hx}{0{,}489}.$$

Integrating this inequality along path $\phi_\sigma(t)$ from $t = t_\sigma^{(5)}$ to $t = T_\sigma$, we have

$$u_\sigma(T_\sigma) - u_\sigma(t_\sigma^{(5)}) > -1{,}1\,(c^2 + b)\, H\varepsilon^2.$$

And thus from (6.113) we can write

$$u_\sigma(T_\sigma) > -1.8(c^2 + b)H\epsilon^2. \tag{6.114}$$

Note at this time that inequalities (6.111) and (6.114) are obtained only when the conditions of either case 1, 2, or 3 are fulfilled.

Lemma 6.4

Let inequalities $a > 0$, $b < 1$, $c^2 + b > 0$ and conditions (6.11)–(6.13) be fulfilled; then the following inequality is fulfilled for any $\sigma \epsilon [0, 1]$:

$$w_\sigma(T_\sigma) > w_\sigma(0). \tag{6.115}$$

Proof

This lemma is proved by contradiction; namely, we suppose that

$$w_\sigma(T_\sigma) \leq w_\sigma(0). \tag{6.116}$$

If supposition (6.116) is fulfilled, then the condition of case 4 could not be fulfilled by the preceding reasoning. Indeed, if we assume that the condition of case 4 is fulfilled, by the definitions of δy_σ and δz_σ, we will have

$$y_\sigma(t_\sigma^{(s)}) = -\eta + \delta y_\sigma \leq -\eta, \tag{6.117}$$

$$z_\sigma(t_\sigma^{(s)}) = -c\eta + \delta z_\sigma \leq -c\eta. \tag{6.118}$$

But $y_\sigma(t)$ and $z_\sigma(t)$ decrease in this case for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$; consequently,

$$y_\sigma(T_\sigma) < -\eta \text{ and } z_\sigma(T_\sigma) < -c\eta. \tag{6.119}$$

However,

$$w_\sigma(T_\sigma) > \frac{1}{2}(1 + c^2)\eta^2 = w_\sigma(0).$$

The last inequality contradicts supposition (6.116). This contradiction thus proves that the conditions of either case 1, 2, or 3 are fulfilled. And then inequalities (6.93), (6.106), (6.111) and (6.114) are also fulfilled.

Introduce the following notations:

$$y_\sigma(T_\sigma) = -\eta + \Delta y_\sigma, \; z_\sigma(T_\sigma) = -c\eta + \Delta z_\sigma. \tag{6.120}$$

Inequalities (6.111) and (6.114) then take on the following forms:

$$-2\eta\Delta y_\sigma - 2c\eta\Delta z_\sigma + \Delta y_\sigma^2 + \Delta z_\sigma^2 > -1.42(1-b)Hc\epsilon^2. \tag{6.121}$$

$$-c\Delta y_\sigma + \Delta z_\sigma > -1.8(c^2+b)H\epsilon^2. \tag{6.122}$$

Inequalities (6.121) and (6.122) permit evaluating the quantity $\frac{dy}{dx} = \frac{z-x}{y-f(x)}$ on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$ more precisely than it was done above by inequality (6.93).

Since inequalities (6.121) and (6.122) are both obtained by the use of inequality (6.93), this represents a more precise estimate of the function $\frac{z-x}{y-f(x)}$ than the estimate by the following approximation.

As for the establishment of inequality (6.93), the following cases can arise.

1. $\Delta y_\sigma \geq 0$, $\Delta z_\sigma \leq 0$.

Inequality (6.93) is true on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$. Integrating this along path $\phi_\sigma(t)$ on this interval, we obtain

$$y_\sigma(t) < y_\sigma(T_\sigma) + 1.38cx_\sigma(t). \tag{6.123}$$

Due to inequality (6.122) and the conditions of this case, we obtain the inequality

$$\Delta y_\sigma < 1{,}8\left(c+\frac{b}{c}\right)H\varepsilon^2, \tag{6.124}$$

$$\Delta z_\sigma > c\Delta y_\sigma - 1.8(c^2+b)H\epsilon^2. \tag{6.125}$$

From relation (6.120) and inequalities (6.123) and (6.125), we obtain

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < \frac{-c\eta + c\Delta y_\sigma - 1{,}8\,(c^2+b)H\varepsilon^2 - x}{-\eta + \Delta y_\sigma + 1{,}38\,cx - cx}$$

or

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < c\,\frac{-\eta + \Delta y_\sigma - 1{,}8\left(c+\frac{b}{c}\right)H\varepsilon^2 - \frac{x}{c}}{-\eta + \Delta y_\sigma + 0{,}38\,cx}.$$

The last inequality can be rewritten in the following way:

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < c + c\,\frac{0{,}38\,cx + \frac{1}{c}x + 1{,}8\left(c+\frac{b}{c}\right)H\varepsilon^2}{\eta - \Delta y_\sigma - 0{,}38\,cx}. \tag{6.126}$$

Since b < 1 by hypothesis from inequality (6.124) and condition (6.7) we obtain

$$\Delta y_\sigma < 0.018.$$

From inequality (6.126), because of the preceding inequality, and from $\eta > 0.7$, there follows the relation

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < c + c\,\frac{0{,}38\ cx + \frac{x}{c} + 1{,}8\left(c+\frac{b}{c}\right)H\varepsilon^2}{0{,}64}$$

or

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < c + c\left[0{,}6\ cx + \frac{1{,}6}{c}x + 3\left(c+\frac{b}{c}\right)H\varepsilon^2\right]. \tag{6.127}$$

This inequality is true for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$ on path $\phi_\sigma(t)$; integrating this we obtain the inequality

$$\begin{aligned} y_\sigma(t) < y_\sigma(T_\sigma) + cx_\sigma(t) + c\,[0{,}3\ cx_\sigma^2(t) + \\ + \frac{0{,}8}{c}x_\sigma^2(t) + 3\left(c+\frac{b}{c}\right)H\varepsilon^2 x_\sigma(t)\Big], \end{aligned} \tag{6.128}$$

which is true on $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$.

Due to the conditions of case 1, relation (6.120) and inequalities (6.124), (6.125) and (6.128), we obtain

$$\begin{aligned} \frac{dy}{dx} = \frac{z-x}{y-f(x)} < \\ < \frac{-c\eta - x - 1{,}8\,(c^2+b)\,H\varepsilon^2}{-\eta + 1{,}8\left(b+\frac{b}{c}\right)H\varepsilon^2 + 0{,}3\,c^2x^2 + 0{,}8\,x^2 + 3\left(c+\frac{b}{c}\right)H\varepsilon^2 x - \alpha(x)} \end{aligned} \tag{6.129}$$

on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$.

2. $\Delta y_\sigma \leq 0$, $\Delta z_\sigma \geq 0$.

Integrating inequality (6.93), we obtain (6.123) and from this, as in the preceding case, the following inequality:

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < \frac{-c\eta - x}{-\eta + 0{,}38\ cx}.$$

Accordingly, it is easy to obtain the inequality

$$\frac{dy}{dx} < c + c\left(0{,}6\ cx + \frac{1{,}6}{c}x\right).$$

From this inequality, by integration we arrive at the relation

$$y_\sigma(t) < y_\sigma(T_\sigma) + cx_\sigma(t) + 0.3c^2x_\sigma^2(t) + 0.8x_\sigma^2(t)$$

for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$ and thus, as in the preceding case, we obtain

$$\frac{dy}{dx} = \frac{z - x}{y - f(x)} < \frac{-c\eta - x}{-\eta + 0.3c^2x^2 + 0.8x^2 - a(x)} \tag{6.130}$$

on path $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$.

3. $\Delta y_\sigma \geq 0$, $\Delta z_\sigma \geq 0$.

As proved above, $y_\sigma(T_\sigma) < 0$; consequently, $\Delta y_\sigma = \eta$. Thus, from (6.121) there follows the inequality

$$\Delta z_\sigma^2 - 2c\eta\Delta z_\sigma > -1.42(1 - b)Hc\epsilon^2.$$

From this inequality one of the two following cases is fulfilled:

$$\Delta z_\sigma > c\eta + \sqrt{c^2\eta^2 - 1.42(1 - b)Hc\varepsilon^2} \tag{6.131}$$

or

$$\Delta z_\sigma < c\eta - \sqrt{c^2\eta^2 - 1.42(1 - b)Hc\varepsilon^2}. \tag{6.132}$$

Consider first the case when inequality (6.131) is fulfilled. Rewrite this inequality in the form

$$\Delta z_\sigma > c\eta + c\sqrt{\eta^2 - 1.42(1 - b)\frac{H}{c}\varepsilon^2}. \tag{6.133}$$

From inequalities $c^2 + b > 0$, (6.7) and (6.22) we obtain

$$(1 - b)\frac{H}{c}\varepsilon^2 < \left(\frac{1}{c} + c\right)H\varepsilon^2 < 0.01.$$

Thus, from (6.133) it is easy to see the inequality

$$\Delta z_\sigma > c\eta + c\left(\eta - 1.1\frac{H}{c}(1 - b)\varepsilon^2\right).$$

Since $\eta > 0.7$, from this inequality we have

$$\Delta z_\sigma > c\eta + c\eta\left(1 - 1.6\frac{H}{c}(1 - b)\varepsilon^2\right).$$

Accordingly, because of (6.7) and (6.22) we obtain

$$\Delta z_\sigma > 1.98c\eta.$$

From this inequality and relation (6.120) we see that

$$z_\sigma(T_\sigma) > 0.98c\eta.$$

However, $z_\sigma(t) \geq z_\sigma(T_\sigma)$ and $x_\sigma(t) \leq \epsilon$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$. Therefore, on the interval of time $t_\sigma^{(s)} \leq t \leq T_\sigma$, there takes place the inequality

$$z_\sigma(t) - x_\sigma(t) > 0.98c\eta - \epsilon.$$

From (6.7) and the definition of η,

$$z_\sigma(t) - x_\sigma(t) > 0.$$

for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$. As proven earlier, this is possible only when $y_\sigma(t) < 0$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$; however,

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < 0 \tag{6.134}$$

on $\phi_\sigma(t)$ for $t\epsilon[t_\sigma^{(s)}, T_\sigma]$. Because of this inequality, inequality (6.129) is also fulfilled when inequality (6.131) is fulfilled.

Return now to the case when inequality (6.132) is fulfilled. From this inequality we easily obtain

$$\Delta z_\sigma^2 - 2c\eta\Delta z_\sigma \leq 0.$$

From this and (6.121) we have

$$\Delta y_\sigma^2 - 2\eta\Delta y_\sigma > -1.42(1-b)Hc\epsilon^2. \tag{6.135}$$

Because of this inequality one of two conditions must be fulfilled:

$$\Delta y_\sigma > \eta + \sqrt{\eta^2 - 1.42(1-b)Hc\varepsilon^2} \tag{6.136}$$

or

$$\Delta y_\sigma < \eta - \sqrt{\eta^2 - 1.42(1-b)Hc\varepsilon^2}. \tag{6.137}$$

Evaluate the quantity $(1-b)Hc\epsilon^2$. If $b \geq 0$, from (6.7) we obtain

$$(1-b)Hc\epsilon^2 < 0.01. \tag{6.138}$$

But if $b < 0$, from (6.3) $H < -\frac{c}{b}$ and then $(1 - b)H < H + c$; thus (6.138) follows from (6.7). From inequalities (6.136) and (6.138) it is not difficult to verify the inequality

$$\Delta y_\sigma > \eta + \eta - 0.0142.$$

Since $y_\sigma(T_\sigma) = \Delta y_\sigma - \eta$, we then have

$$y_\sigma(T_\sigma) > \eta - 0.0142.$$

Because $\eta > 0.7$, from the last inequality it follows that $y_\sigma(T_\sigma) > 0$, and this, as mentioned above, is not true. Therefore, inequality (6.136) cannot be realized; consequently, inequality (6.137) is fulfilled. From inequality (6.137) with the aid of (6.138), it is not difficult to obtain the inequality

$$\Delta y_\sigma < 1.1(1 - b)Hc\epsilon^2. \tag{6.139}$$

Inequality (6.93) is fulfilled on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$. Integrating this along $\phi_\sigma(t)$ on interval $t_\sigma^{(s)} \leq t \leq T_\sigma$, we obtain inequality (6.123). Moreover, inequality (6.125) is true in this case as in case 1. From inequalities (6.123) and (6.125) we obtain

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < c\frac{-\eta + \Delta y_\sigma - 1{,}8\left(c + \frac{b}{c}\right)H\varepsilon^2 - \frac{x}{c}}{-\eta + \Delta y_\sigma + 0{,}38\, cx}.$$

Following from inequalities (6.138) and (6.139) we have

$$\Delta y_\sigma < 0.011.$$

Thus, in this case inequalities (6.127) and (6.128) are true. As in case 1, from inequalities (6.128) and (6.139), we obtain the following inequality which is true on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$:

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)}$$

$$< \frac{-c\eta - x}{-\eta + 1{,}1\,(1-b)\,Hc\varepsilon^2 + 0{,}3c^2x^2 + 0{,}8\,x^2 + 3\left(c + \frac{b}{c}\right)H\varepsilon^2 x - \alpha(x)}. \tag{6.140}$$

From cases 1, 2, and 3 and relations (6.4) and (6.5), the inequality

$$\frac{dy}{dx} = \frac{z-x}{y-f(x)} < \frac{c\eta - x - 1{.}8\,(c^2 + b)\,H\varepsilon^2}{\eta + \alpha(x) - \mu\varepsilon^2}. \tag{6.141}$$

is fulfilled in these cases on path $\phi_\sigma(t)$ for $t \epsilon [t_\sigma^{(s)}, T_\sigma]$.

4. $\Delta y_\sigma \le 0$, $\Delta z_\sigma \le 0$.

We will show that one of these inequalities is strictly fulfilled; if not, we would arrive at the conditions of one of the preceding cases. This case cannot be realized. Indeed, if the conditions of the present case were fulfilled, we would have

$$w_\sigma(T_\sigma) = \frac{1}{2} y_\sigma^2(T_\sigma) + \frac{1}{2} z_\sigma^2(T_\sigma) > \frac{1}{2}\eta^2(1 + c^2) = w_\sigma(0),$$

which contradicts supposition (6.116). Thus, if supposition (6.116) is fulfilled, inequality (6.141) is also fulfilled.

Now evaluate the quantity $\frac{dy}{dx} = \frac{z - x}{x - f(x)}$ on the interval of time $0 \le t \le t_\sigma^{(1)}$ for the paths $\phi_\sigma(t)$ for which $0.5 \le \sigma \le 1$. Go first to path $\phi_1(t)$ and evaluate $y_1(t)$ on interval $[0, t_1^{(1)}]$. From equality (3.7) we see that

$$\frac{dy}{dx} = \frac{z - x}{y - f(x)} < \frac{z}{y - f(x)}.$$

Since $z_\sigma(t)$ decreases and $y_\sigma(t)$ increases with increasing time for $t \epsilon [0, t_\sigma^{(1)}]$, from the last inequality we obtain the following estimate which is true on $\phi_1(t)$ for $t \epsilon [0, t_1^{(1)}]$:

$$\frac{dy}{dx} < \frac{ck}{k - cx - Hx} = c + c\frac{cx + Hx}{k - cx - Hx}.$$

However, $x_1(t) \le \epsilon$ on interval of time $0 \le t \le t_1^{(1)}$; therefore, from the last inequality, because of (6.7) we obtain

$$\frac{dy}{dx} < c + 1.2c(cx + Hx).$$

Integrating this inequality on the interval of time $0 \le t \le t_1^{(1)}$, we have

$$y < k + cx + 0.6c(c + H)x^2. \tag{6.142}$$

This inequality is proved for path $\phi_1(t)$; however, according to lemma 3.9, it will also be true for all paths $\phi_\sigma(t)$ on the interval of time $0 \le t \le t_\sigma^{(1)}$.

Now evaluate $z_{0.5}(t)$ for $t \epsilon [0, t_{0.5}^{(1)}]$. Since $y_{0.5}(t)$ increases along with time on this interval, for $\phi_{0.5}(t)$ and $t \epsilon [0, t_{0.5}^{(1)}]$, we will have

$$\frac{dz}{dx} = \frac{-cx - b\alpha(x)}{y - f(x)} > -\frac{cx + bHx}{0.5 - c\varepsilon - H\varepsilon} > -\frac{cx + bHx}{0.4}.$$

Integrating this inequality, we obtain

$$z > ck - 1.25(c + bH)x^2. \tag{6.143}$$

The last inequality is proved only for path $\phi_{0.5}(t)$; however, because of lemma 3.9 it is true for all paths $\phi_\sigma(t)$ for which $0.5 \le \sigma \le 1$ for $t\epsilon[0, t_\sigma^{(1)}]$.

We will consider the functions $y_\sigma(t)$, $z_\sigma(t)$ and $w_\sigma(t)$ on path $\phi_\sigma(t)$ for $t\epsilon[0, t_\sigma^{(1)}]$ and $t\epsilon[t_\sigma^{(5)}, T_\sigma]$ as functions of the x component. In this case, multiple values arise since path $\phi_\sigma(t)$ passes through strip $0 \le x \le \epsilon$ twice for interval $[0, T_\sigma]$. In order to avoid this multiple value, we will index our function with the sign + on the interval of time $[0, t_\sigma^{(1)}]$ and with the sign – on the interval $[t_\sigma^{(5)}, T_\sigma]$. Because of equality (6.32) and inequality (6.141) we have

$$\frac{dw_-}{dx} < (1-b)\frac{c\eta + x + 1.8(c^2+b)H\varepsilon^2}{\eta + \alpha(x) - \mu\varepsilon^2}\alpha(x) \tag{6.144}$$

on all paths $\phi_\sigma(t)$. On the other hand, from equality (6.32) and inequalities (6.142) and (6.143) we obtain

$$\frac{dw_+}{dx} > (1-b)\frac{ck - x - 1.25(c+bH)x^2}{k + 0.6c(c+H)x^2 - \alpha(x)}\alpha(x). \tag{6.145}$$

The last inequality is true for those paths $\phi_\sigma(t)$ for which $0.5 \le \sigma \le 1$. Subtracting inequality (6.144) from inequality (6.145), we obtain

$$\begin{aligned}\frac{dw_+}{dx} - \frac{dw_-}{dx} > (1-b)\Big[&\frac{ck - x - 1.25(c+bH)x^2}{k + 0.6c(c+H)x^2 - \alpha(x)} \\ &- \frac{c\eta + x + 1.8(c^2+b)H\varepsilon^2}{\eta + \alpha(x) - \mu\varepsilon^2}\Big]\alpha(x).\end{aligned} \tag{6.146}$$

In the last inequality we will combine the fraction in the brackets and designate its numerator by A and its denominator by B. Then we can write

$$\begin{aligned}A = c(k+\eta)(\alpha(x) - x) - ck\mu\varepsilon^2 - 1.8k(c^2+b)H\varepsilon^2 \\ - 1.25\eta(c+bH)x^2 - 0.6c^2\eta(c+H)x^2 - 1.25(c+bH)x^2\alpha(x) \\ + \mu\varepsilon^2 x + 1.8(c^2+b)H\varepsilon^2\alpha(x) - 0.6c(c+H)x^3 \\ + 1.25(c+bH)\mu\varepsilon^2 x^2 - 0.6\cdot 1.8(c^2+b)(c+H)cH\varepsilon^2 x^2\end{aligned} \tag{6.147}$$

and

$$B = [k + 0.6c(c+H)x^2 - \alpha(x)][\eta + \alpha(x) - \mu\epsilon^2]. \tag{6.148}$$

Estimate A for $x\epsilon[0, \delta]$. Using (6.7), from (6.147) we easily obtain

$$\begin{aligned}A > c(k+\eta)\Big[(\alpha(x) - x) - \mu\varepsilon^2 - 1.8\left(c + \frac{b}{c}\right)H\varepsilon^2 \\ - 0.625\left(1 + \frac{bH}{c}\right)x^2 - 0.35c(c+H)x^2\Big].\end{aligned}$$

Due to this inequality and inequalities (6.2), (6.3) and (6.8), we have

$$A > c(k+\eta)\left[0.9\left(h - \frac{1}{c}\right)x - \mu\varepsilon^2 - 1.8\left(c + \frac{b}{c}\right)H\varepsilon^2\right]. \tag{6.149}$$

Thus, from inequality (6.9) we come to

$$A > 0{,}9\,c\,(k+\eta)\left(h-\frac{1}{c}\right)(x-0{,}2\delta). \tag{6.150}$$

From equality (6.148) and inequalities (6.7) and (6.11) it is easy to verify the following estimate:

$$B < 1.15k\eta.$$

Therefore, from (6.150) we obtain

$$\frac{dw_+}{dx}-\frac{dw_-}{dx} > \frac{0{,}9\cdot 2c}{1{,}15\,k}(1-b)\left(h-\frac{1}{c}\right)(x-0{,}2\delta)\,hx. \tag{6.151}$$

Designate by $\Delta_1 w_\sigma$ the increasing function $w_\sigma(t)$ for a passage of path $\phi_\sigma(t)$ in strip $0.2\delta \leq x \leq \delta$. Integrating (6.151), after simplifying the expression we obtain

$$\Delta_1 w_\sigma > \frac{0{,}35}{k}\,c\,(1-b)\left(h-\frac{1}{c}\right)h\delta^3. \tag{6.152}$$

This inequality was discovered from inequalities (6.144) and (6.145). But inequality (6.145) is true only for those paths $\phi_\sigma(t)$ for which $0.5 \leq \sigma \leq 1$; consequently, inequality (6.152) is also true only for $0.5 \leq \sigma \leq 1$.

Consider now those paths $\phi_\sigma(t)$ for which $0 \leq \sigma \leq 0.5$. Estimate $\frac{dy}{dx}=\frac{z-x}{y-f(x)}$ for these paths for $t\epsilon[0,\, t_\sigma^{(1)}]$. Evaluating $y_{0.5}(t)$ for $t\epsilon[0,\, t_{0.5}^{(1)}]$, we have

$$\frac{dy}{dx}=\frac{z-x}{y-f(x)} < \frac{ck}{0.5k-cx-Hx} = 2c + \frac{2c\,(c+H)\,x}{0{,}5k-cx-Hx}.$$

Thus, because of inequality (6.7) we obtain

$$\frac{dy}{dx} < 2c + \frac{2c\,(c+H)\,x}{0{,}4}.$$

Integrating this inequality along path $\phi_{0.5}(t)$, we see that

$$y < 2cx + 2.5c(c+H)x^2 + 0.5k. \tag{6.153}$$

This inequality is proved only for $\phi_{0.5}(t)$; however, due to lemma 3.9 it is true for all paths $\phi_\sigma(t)$ for which $0 \leq \sigma \leq 0.5$. From inequality (6.25) and (6.153) we have

$$\frac{dy}{dx}=\frac{z-x}{y-f(x)} > \frac{0{,}9\,ck-x}{0.5\,k+cx+2.5\,c\,(c+H)\,x^2-\alpha(x)}.$$

Since inequality (6.11) takes place for $0 \le x \le \delta$, from the last inequality by virtue of (6.7) we obtain

$$\frac{dy}{dx} > c\,\frac{0{,}9\,k - \frac{x}{c}}{0{,}625 - \frac{x}{c}} > 1{,}44\,c. \tag{6.154}$$

This inequality is true for all paths $\phi_\sigma(t)$ for which $0 \le \sigma \le 0.5$ for $t \in [0, t_\sigma^{(1)}]$. From inequality (6.154) and equality (6.32), for $0 \le \sigma \le 0.5$

$$\frac{dw_+}{dx} > 1{,}44\,c\,(1-b)\,\alpha(x). \tag{6.155}$$

On the other hand, from inequality (6.144) which is true for any $\sigma \in [0, 1]$, we see that

$$\frac{dw_-}{dx} < c\,(1-b)\,\frac{\eta + \frac{x}{c} + 0{,}018}{\eta + \frac{x}{c} - 0{,}03}\,\alpha(x).$$

Thus, by virtue of $\eta > 0.7$, we obtain

$$\frac{dw_-}{dx} < 1{,}1\,c\,(1-b)\,\alpha(x). \tag{6.156}$$

From inequalities (6.155) and (6.156) we have

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > 0{,}3\,c\,(1-b)\,hx. \tag{6.157}$$

This inequality is true for all paths $\phi_\sigma(t)$ for which $0 \le \sigma \le 0.5$ for $0 \le x \le \delta$.

Designate by $\Delta_2 w_\sigma$ the increase of function $w_\sigma(t)$ for the crossing of path $\phi_\sigma(t)$ on strip $0 \le x \le \delta$. Integrating inequality (6.157) results in

$$\Delta_2 w_\sigma > 0.15c(1-b)h\delta^2.$$

Thus, from (6.7) we obtain

$$\Delta_2 w_\sigma > 1{,}5\,c\,(1-b)\,h\left(h - \frac{1}{c}\right)\delta^3, \tag{6.158}$$

this inequality is true only for $0 \le \sigma \le 0.5$.

Consider now the increase of function $w_\sigma(t)$ for the passage of path $\phi_\sigma(t)$ on strip $0 \le x \le 0.2\delta$. Designate this increase by $\Delta_3 w_\sigma$ so that $\Delta_2 w_\sigma = \Delta_1 w_\sigma + \Delta_3 w_\sigma$. For this we will consider only those paths $\phi_\sigma(t)$ for which $0.5 \le \sigma \le 1$. From inequality (6.149), true exactly for such σ,

$$A > -c(k+\eta)\left[\mu + 1{,}8\left(c+\frac{b}{c}\right)H\right]\varepsilon^2. \tag{6.159}$$

From equality (6.148) and inequality (6.7), we see that

$$B > (k - \alpha(x))(\eta - \mu\epsilon^2) > 0.6(k - 0.1). \tag{6.160}$$

Consequently, for $0 \le x \le 0.2\delta$ we have

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > -\frac{c(k+\eta)\left[\mu + 1{,}8\left(c+\frac{b}{c}\right)H\right]}{0{,}6(k-0{,}1)}(1-b)\varepsilon^2\alpha(x). \tag{6.161}$$

Integrating the last inequality and using inequality $\eta \le k$, we obtain

$$\Delta_3 w_\sigma > -0{,}08\, ck\,\frac{(1-b)H}{k-0{,}1}\left[\mu + 1{,}8\left(c+\frac{b}{c}\right)H\right]\varepsilon^2\delta^2. \tag{6.162}$$

Thus, from (6.152) we write

$$\begin{aligned} \Delta_2 w_\sigma > \frac{0{,}35}{k}c(1-b)\left(h-\frac{1}{c}\right)h\delta^3 \\ -0{,}08\, ck\,\frac{(1-b)H}{k-0{,}1}\left[\mu + 1{,}8\left(c+\frac{b}{c}\right)H\right]\varepsilon^2\delta^2. \end{aligned} \tag{6.163}$$

The last inequality has been proved only for $0.5 \le \sigma \le 1$. But from (6.158) it is true for all $\sigma\epsilon[0.1]$.

Now estimate the increase of function $w_\sigma(t)$ for the passage of path $\phi_\sigma(t)$ on strip $\delta \le x \le \epsilon$. Designate this increase by $\Delta_4 w_\sigma$. From inequality (6.147) and inequality (6.7), the following inequality is obtained:

$$A > -2c(k+\eta)\epsilon.$$

Thus, from (6.160) we find the relation

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > -\frac{c(k+\eta)\varepsilon}{0{,}3(k-0{,}1)}(1-b)\alpha(x).$$

is fulfilled for $\delta \le x \le \epsilon$. Accordingly, from (6.12) we have

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > -\frac{2ckH\varepsilon}{0{,}3(k-0{,}1)}(1-b)x.$$

Integrating this inequality from $x = \delta$ to $x = \epsilon$ and using (6.6), we obtain

$$\Delta_4 w_\sigma > -\frac{20ckH}{3(k-0{,}1)}(1-b)\varepsilon^2\delta^2. \tag{6.164}$$

Estimate finally the increase of function $w_\sigma(t)$ on the interval of time $t_\sigma^{(1)} \leq t \leq t_\sigma^{(5)}$. Designate this increase by $\Delta_5 w_\sigma$. Since $w_\sigma(t)$ increases for $t \epsilon [t_\sigma^{(1)}, t_\sigma^{(2)}]$, $\Delta_5 w_\sigma$ is larger than the increase of $w_\sigma(t)$ on the interval of time $t_\sigma^{(2)} \leq t \leq t_\sigma^{(5)}$. If it results that $t_\sigma^{(5)} > t_\sigma^{(6)}$, then $\Delta_5 w_\sigma$ is larger than the increase of $w_\sigma(t)$ on interval $[t_\sigma^{(2)}, t_\sigma^{(6)}]$. However, inequality (6.37) is true for $x \geq \epsilon$. Integrating this inequality and using (6.39) we obtain

$$\Delta_5 w_\sigma > -2(1-b)\sqrt{k^2(1+c^2)+0.012\,\delta^4}. \tag{6.165}$$

From inequalities (6.163)–(6.165) and condition (6.10) we obtain

$$w_\sigma(T_\sigma) - w_\sigma(0) > 0.$$

The last inequality contradicts supposition (6.116), and the contradiction obtained proves the lemma.

Lemma 6.5

Let inequalities $a > 0$ and $b < 1$ be fulfilled; moreover, in a neighborhood of point $x = 0$, let function $\alpha(x)$ be differentiable and $\frac{d\alpha}{dx} > 0$. Then, if point p lies in plane $x = 0$ sufficiently close to the origin and if $t_1 > 0$ is such that point $\phi(p, t_1)$ also lies in plane $x = 0$, there occurs inequality

$$w(p) > w(\phi(p, t_1)). \tag{6.166}$$

The assertion of this lemma results because the null solution of system (2.15) is stable in the sense of Lyapunov and because of equalities (4.98) and (4.99).

Consider a path $\phi(m, t)$ of system (2.15), with its initial point m lying on half-line $\{x = 0, z = cy, y > 0\}$. Let T_m be the first instant after $t = 0$ of the intersection of path $\phi(m, t)$ with plane $x = 0$. If it happens that $x > 0$ for $t > 0$ on path $\phi(m, t)$, we will say that $T_m = +\infty$. The following lemma is true.

Lemma 6.6

Let inequalities $a > 0$, $b < 1$, $c^2 + b > 0$ be fulfilled. Then for all $t \epsilon (0, T_m)$ there is fulfilled inequality

$$u(\phi(m, t)) < 0 \tag{6.167}$$

where u is the function introduced by equality (6.68).

Proof

Since point m lies on line $\{x = 0, z - cy\}$, $u = 0$ on path $\phi(m, t)$ for $t = 0$. Thus, from equality (6.69) we obtain

$$u = -(c^2 + b)e^{-ct}\int_0^t \alpha(x)\cdot e^{ct}dt \tag{6.168}$$

on path $\phi(m, t)$. Since it results that $x > 0$ and $\alpha(x) > 0$ on the interval of time $(0, T_m)$ on path $\phi(m, t)$, from the integral relation (6.168) it follows that inequality (6.167) is true on the interval of time $0 < t \leq T_m$.

Using the lemmas proved, it is not difficult to verify the following theorem.

Theorem 6.1

Suppose that conditions $a > 0$ and $0 \leq b < 1$ are fulfilled; suppose, moreover, that inequality $c \geq 1$ is fulfilled. Let conditions (6.11)–(6.13) be fulfilled; furthermore, in a neighborhood of point $x = 0$, let function $\alpha(x)$ be differentiable and $\frac{d\alpha}{dx} > 0$. Finally, for $|x| \leq \frac{1}{c}\sqrt{k^2(1 + c^2) + 0.012}$, let there take place the inequality

$$f(-x) = -f(x). \tag{6.169}$$

Then system (2.15) has a periodic solution.

Proof

In plane $x = 0$, consider a point q_0 with coordinates $x = 0$, $y = 0$, $z = c\beta$; point q_1 with coordinates $x = 0$, $y = \beta$, $z = c\beta$; and segment M of the straight line $\{x = 0, z = c\beta\}$, included between points q_0 and q_1. Suppose that $0 < \beta < 1$ and β is so small that inequality (6.166) is fulfilled on the entire segment M. The existence of such a β is guaranteed by lemma 6.5.

Now consider a trapezoid $p_0p_1q_0q_1p_0$ in plane $x = 0$. We will consider all paths $\phi(1, t)$ of system (2.15), the initial points of which lie inside or on the boundary of this trapezoid. Let $T_l > 0$ be the first instant after $t = 0$ of the intersection of path $\phi(1, t)$ with plane $x = 0$. By theorem 3.2, instant T_l exists and is finite. On plane yOz of the phase space, introduce polar coordinates by means of equalities $y = r\cos\theta$ and $z = r\sin\theta$. On the trapezoid we give the following two functions of its points:

$$\Delta r(l) = r(\phi(l, T_l)) - r(l), \tag{6.170}$$

$$\Delta\theta(l) = \theta(\phi(l, T_l)) - \theta(l) - \pi. \tag{6.171}$$

By $r(l)$ and $\phi(l)$ are meant the radius vector and the polar angle of point l. From theorem 3.3, the reasoning of section 3 and the theorem on integral continuity, functions Δr and $\Delta\theta$ are continuous.

From lemma 6.4 it follows that

$$\Delta r(p) > 0, \tag{6.172}$$

when point p lies on side p_0p_1 of our trapezoid.

In consequence of inequality (6.166),

$$\Delta r(q) < 0, \tag{6.173}$$

when point q lies on side q_0q_1 of the trapezoid.

Further, from lemma 6.6 it is not difficult to verify that

$$\Delta\theta(m) > 0, \tag{6.174}$$

when point m lies on segment p_1q_1 of the straight line $\{x = 0,\ z = cy\}$.

Finally, let point n lie on segment p_0q_0 of axis Oz, then from theorem 3.3 we have

$$\Delta\theta(n) < 0. \tag{6.175}$$

From inequalities (6.172)–(6.175) and the continuity of functions Δr and $\Delta\theta$, there exists a point l_0 inside the trapezoid $p_0p_1q_0q_1p_0$ such that

$$\Delta r(l_0) = \Delta\theta(l_0) = 0. \tag{6.176}$$

Consequently, points $\phi(l_0, T_{l_0})$ are symmetrical to point l_0 relative to the origin.

Note now that from lemma 3.9 and inequality (6.40) all paths $\phi(l, t)$ for $t \epsilon [0, T_l]$ lie in the strip $|x| < \frac{1}{c}\sqrt{k^2(1 + c^2) + 0.012}$. But in this strip, as shown by condition (6.169) of the theorem to be proven, the field of linear elements defined by system (2.15) is symmetric relative to the origin. Therefore, path $\phi(l_0, t)$ intersects plane $x = 0$ at point l_0 at a time following $t = T_{l_0}$, and thus $\phi(l_0, t)$ is a path of a periodic motion of system (2.15).

The theorem is proved.

Corollary

Suppose that $a > 0$, $0 \leq b < 1$ and $c \geq 1$. Suppose further that

$$hx \leq \alpha(x) \leq Hx \text{ for } 0 \leq x \leq K\delta, \tag{6.177}$$

$$0 < \alpha(x) \leq Hx \text{ for } K\delta \leq x \leq K\epsilon, \tag{6.178}$$

$$0 < \alpha(x) < K\delta^4 \quad \text{for} \quad K\varepsilon \leqslant x \leqslant \frac{K}{c}\sqrt{k^2(1 + c^2) + 0.012}, \tag{6.179}$$

where the numbers h, H, δ and ϵ satisfy inequalities (6.2), (6.3), (6.6)–(6.10) and K is an arbitrary positive number. Moreover, in a neighborhood of point $x = 0$, let function $\alpha(x)$ be differentiable and $d\alpha/dx > 0$.

Finally, let equality (6.169) take place for $|x| \leq \frac{K}{c}\sqrt{k^2(1 + c^2) + 0.012}$. Then system (2.15) has a periodic motion. For proof, the change of variables $x = Kx_2$, $y = Ky_2$ and $z = Kz_2$ should be made and theorem 6.1 should be applied.

Section 19

In this section we will consider those cases which are not covered in theorem 6.1, i.e., the cases when $a > 0$, $0 \leq b < 1$, $c < 1$ and when $a > 0$, $b < 0$, and $c^2 + b > 0$. Here we will show that periodic motions can also appear for system (2.15). In these cases, the proof of theorem 6.1 does not proceed because we cannot prove the continuity of functions $\Delta r(l)$ and $\Delta\theta(l)$ introduced by equalities (6.170) and (6.171). The continuity of these functions can be violated in spite of the theorem on the continuous dependence of the solutions on the initial conditions.

Indeed, suppose that in trapezoid $p_0p_1q_1q_0p_0$ there exists a point p such that path $\phi(p, t)$ proceeds in the following way. There exists a point $T_0 > 0$ such that $x > 0$ for $t\epsilon(0, T_0)$ on $\phi(p, t)$ and

$$x(\phi(p, T_0)) = 0, \; y(\phi(p, T_0)) = 0, \; z(\phi(p, T_0)) > 0.$$

Then from the reasoning of section 3, path $\phi(p, t)$ is tangent to plane $x = 0$ for $t = T_0$ in such a way that it lies in half-space $x > 0$ for $t \neq T_0$ and sufficiently close to T_0. It is not difficult to see that at point p the continuity of functions Δr and $\Delta\theta$ are violated. We will analyze in detail the behavior of these trajectories.

In this section as in the preceding, we will suppose that function $\alpha(x)$ satisfies conditions (6.11)–(6.13). In addition we will suppose that function $\alpha(x)$ also satisfies the following conditions. We will assume that $\alpha(x)$ is differentiable for $0 \leq x \leq \delta$ and that

$$\frac{d\alpha}{dx} > 0 \text{ for } 0 \leq x < \delta. \tag{6.180}$$

Moreover, we will suppose that

$$\alpha(x) < \alpha(\delta) \text{ for } \delta < x \leq \epsilon. \tag{6.181}$$

Also in this section, we will suppose that inequalities (6.2), (6.3) and (6.6)–(6.10) are fulfilled. Assume

$$\nu = \sqrt{k^2(1 + c^2) + 0.012} \tag{6.182}$$

and

$$\zeta = \begin{cases} (c + bH)(c + h)\delta & \text{for } b < 0 \\ c(c + h)\delta & \text{for } b \geq 0. \end{cases} \tag{6.183}$$

Relative to numbers δ and ϵ we will still suppose that there are fulfilled the inequalities

$$\varepsilon - \delta < \min\left\{\frac{\zeta^2}{20\nu c}, \frac{\zeta^4}{20\nu^2\varepsilon}, \frac{1}{c}\delta^2\right\} \tag{6.184}$$

and

$$2(1-b)\nu\delta^4 < \left(\frac{1}{2} - \frac{0{,}03}{\nu}\right)\zeta^2. \tag{6.185}$$

Now let point p lie inside or on the boundary of trapezoid $p_0p_1q_1q_0p_0$. Suppose that an instant of time T_0 exists such that $x > 0$ for $t \epsilon (0, T_0)$ on path $\phi(p, t)$ and relations $x = 0$, $y = 0$ and $z > 0$ take place for $t = T_0$ on $\phi(p, t)$. Then path $\phi(p, t)$ is tangent to surface $x = 0$ for $t = T_0$. Since z decreases for $t \epsilon [0, T_0]$ along path $\phi(p, t)$ and since $y(\phi(p, T_0)) = 0$, we have

$$w(p) > w(\phi(p, T_0)), \tag{6.186}$$

where w is the function defined by equality (5.38).

Suppose that path $\phi(p, t)$ intersects plane $x = 0$ for $t = \tau > 0$. For this we will consider that τ is the first instant after $t = 0$ of the intersection of $\phi(p, t)$ with plane $x = 0$, i.e., we will consider that on path $\phi(p, t)$ there take place the relations

$$x(t) \geq 0 \text{ for } t \epsilon [0, \tau],\ x(\tau) = 0,\ y(\tau) < 0. \tag{6.187}$$

We will show there is fulfilled the inequality

$$w(p) > w(\phi(p, \tau)). \tag{6.188}$$

From the reasoning of chapter III, path $\phi(p, t)$ must proceed in the following way. For an increase of time t from 0 it remains inside domain $\{x > 0,\ y - f(x) > 0,\ z - x > 0\}$ until it does not intersect plane $z - x = 0$ for $t = t_1$. Further, path $\phi(p, t)$ intersects surface $y - f(x) = 0$ and goes into domain $\{x > 0,\ y - f(x) < 0,\ z - x < 0\}$ for $t = t_2 \geq t_1$. Since path $\phi(p, t)$ is tangent to plane $x = 0$ for $t = T_0$ and since $z(\phi(p, T_0)) > 0$ for this, then obviously a $t_4 > t_2$ can be found such that path $\phi(p, t)$ intersects plane $z - x = 0$ for $t = t_4$. In this case, if $y(t_4) < 0$ on path $\phi(p, t)$, there exists a $t_3 \epsilon (t_2, t_4)$ such that on $\phi(p, t)$

$$y(t_3) = 0, \tag{6.189}$$

i.e., for $t = t_3$, path $\phi(p, t)$ intersects plane $y = 0$ and goes into the half space of the negative y component. If the instant of time t_3 exists (i.e., if $y(t_4) < 0$), then path $\phi(p, t)$ must, for time increasing from $t = t_4$, intersect plane $y = 0$ one more time for $t = t_5$ (so that $y < 0$ for $t \epsilon (t_3, t_5)$ on $\phi(p, t)$). In this case it can happen that $t_5 = T_0$, and then path $\phi(p, t)$ touches plane $x = 0$ after two intersections with plane $z - x = 0$. It can also happen that $t_5 < T_0$, and then $\phi(p, t)$ touches plane $x = 0$ after a larger number of intersections with plane $z - x = 0$. (Note that the number of intersections of path $\phi(p, t)$ with plane $z - x = 0$ on the interval of time $0 < t < T_0$ must be even.) In this way we have

$$0 < t_1 \leq t_2 < t_3 < t_4 < t_5 \leq T_0. \tag{6.190}$$

Analogous to the preceding, designate as T_1 the first instant after $t = T_0$ of the intersection of path $\phi(p, t)$ with plane $z - x = 0$ and designate as T_2 the first instant after $t = T_0$ of the intersection of $\phi(p, t)$ with surface $y - f(x) = 0$ so that $T_0 < T_1 \leq T_2$.

Now consider the points of intersection of path $\phi(p, t)$ with plane $x = 0$; i.e., point $\phi(p, \tau)$. We begin to move along path $\phi(p, t)$ from point $\phi(p, \tau)$ in the direction of decreasing time. If $z(\phi(p, \tau)) > 0$, then we will first be in domain $\{x > 0, y < 0, z - x > 0\}$; for $t = \tau_1 < \tau$, path $\phi(p, t)$ intersects plane $z - x = 0$ and goes into domain $\{x > 0, y < 0, z - x < 0\}$. In this domain, along all motions of system (2.15), y increases with decreasing time; consequently, a $\tau_3 < \tau_4$ can be found such that $\phi(p, t)$ intersects plane $y = 0$ and goes into domain $\{x > 0, 0 < y < f(x), z - x < 0\}$ for $t = \tau_3$. Finally, path $\phi(p, t)$ intersects either surface $y - f(x) = 0$ or plane $z - x = 0$ for $t = \tau_2 < \tau_3$. One of two things is possible: on path $\phi(p, t)$ it results that either

$$y(\tau_2) - f(x(\tau_2)) = 0,\ z(\tau_2) - x(\tau_2) < 0, \tag{6.191}$$

or

$$y(\tau_2) - f(x(\tau_2)) \leq 0,\ z(\tau_2) - x(\tau_2) = 0. \tag{6.192}$$

If the possibility characterized by relation (6.191) is realized, then for additional decreases of time, path $\phi(p, t)$ intersects plane $z - x = 0$ for $t = \tau_1 < \tau_2$ and goes into domain $\{x > 0, y > f(x), z - x > 0\}$. Finally path $\phi(p, t)$ intersects surface $y - f(x) = 0$ for $t = \tau_0 < \tau_1$. In this case, it can happen that $T_0 = \tau_0$; it can also result that $T_0 < \tau_0$.

In the following we will often consider functions y, z, w and t on path $\phi(p, t)$ as functions of abscissa x. In order to avoid multiple values arising for this, introduce the following designations. On interval $0 \leq t \leq t_2$ of path $\phi(p, t)$ we will give these functions the index 1 and the sign +, for example, $y_1^+(x)$. On interval $t_2 \leq t \leq t_5$ of path $\phi(p, t)$, identify the functions with the index 1 and the sign −, for example, $z_1^-(x)$. (The instant of time t_5 also cannot be defined; this happens if $y(\phi(p, t_4)) \geq 0$; then this function is not considered on interval $t_2 \leq t \leq T_0$.) On interval $T_0 \leq t \leq T_2$, it is given the index 2 and the sign +, for example, $w_2^+(x)$. Further, on interval $\tau_0 \leq t \leq \tau_2$ (only if instants τ_0 and τ_1 are defined, i.e., if relations (6.191) are realized) identify the functions with the index 3 and the sign +, for example, $t_3^+(x)$; and on interval $\tau_2 \leq t \leq \tau$, the index 3 and the sign −. Since, in the segments considered, x varies monotonically along path $\phi(p, t)$, with these designations we completely preserve the single valuedness. /143

Further, the following cases can occur:

I. On path $\phi(p, t)$, $x(\tau_2) \leq \delta$.

II. On path $\phi(p, t)$, $x(\tau_2) > \delta$.

We will begin the proof of inequality (6.188) with case II. Evaluate in this case the difference $z(p) - z(\phi(p, T_0))$. According to lemma 3.9 we have

$$x(t_2) > x(\tau_2) > \delta \tag{6.193}$$

on path $\phi(p, t)$. But for $t \epsilon (0, t_2)$ on path $\phi(p, t)$ there is fulfilled the inequality

$$y \geq f(x), \tag{6.194}$$

Thus from (6.193) there follows inequality

$$y_1^+(\delta) > f(\delta). \tag{6.195}$$

For $t = \vartheta > 0$, let path $\phi(p, t)$ be the first time after $t = 0$ of the intersection of plane $y = 0$. It is not difficult to see that $\vartheta < T_0$. By M designate the set of those values $t \epsilon [0, \vartheta]$ for which on path $\phi(p, t)$ there is fulfilled the inequality

$$\frac{dy}{dt} = z - x < 0. \tag{6.196}$$

From the definition of ϑ, $z(\phi(p, t)) > 0$ since $z(\phi(p, T_0)) > 0$ for $t \epsilon [0, \vartheta]$, and z decreases along $\phi(p, t)$ with increasing time for $t \epsilon [0, T_0]$.

From equality $\frac{dz}{dy} = \frac{-cx - b\alpha(x)}{z - x}$ it follows that on path $\phi(p, t)$ for $t \epsilon M$ there is fulfilled the inequality

$$\frac{dz}{dy} > c + b\frac{\alpha(x)}{x}. \tag{6.197}$$

From this inequality and conditions of (6.11)–(6.13),

$$\frac{dz}{dy} > \frac{\zeta}{(c + h)\delta} \tag{6.198}$$

on path $\phi(p, t)$ for $t \epsilon M$. Multiplying this inequality by dy/dt, we obtain the following inequality true for $t \epsilon M$:

$$\frac{dz}{dt} < \frac{\zeta}{(c + h)\delta}\frac{dy}{dt}.$$

Integrating this inequality along path $\phi(p, t)$ on set M we obtain

$$\int_M \frac{dz}{dt} dt < \frac{\zeta}{(c + h)\delta}\int_M \frac{dy}{dt} dt. \tag{6.199}$$

However, on the interval of time $[0, \vartheta]$, points are present in which inequality (6.195) is fulfilled for t = and y = 0 on $\phi(p, t)$. Therefore, the integral standing on the right in inequality (6.199) is less than $-f(\delta)$. But from relation (3.29) and condition (6.11), $f(\delta) \geq c\delta + h\delta$; therefore, we have

$$\int_{M} \frac{dz}{dt}\,dt < -\zeta.$$

Since $\frac{dz}{dt} \le 0$ for all $t\epsilon[0, \vartheta]$, from the last inequality we obtain

$$\int_{0}^{\vartheta} \frac{dz}{dt}\,dt < -\zeta$$

or

$$z(\phi(p, \vartheta)) - z(p) < -\zeta.$$

Because $\vartheta < T_0$, from the last inequality we see that

$$z(p) - z(\phi(p, T_0)) > \zeta. \tag{6.200}$$

Inequality (6.200) is true only in the conditions of case II.

One of two possibilities can exist in the conditions of case II: II_1, relation (6.191) is realized; II_2, relation (6.192) is realized.

We begin by considering case II_1. In this case the instants τ_0 and τ_1 are defined as instants of the intersection of path $\phi(p, t)$ with surface $y - f(x) = 0$ and plane $z - x = 0$ respectively.

For case II_1 we will distinguish between the following possibilities:

1.*	$x(\tau_2) \in (\delta, \varepsilon)$	$x(t_1) \leqslant \delta$,
2.*	$x(\tau_2) \in (\delta, \varepsilon)$	$x(t_1) \in (\delta, \varepsilon)$,
3.*	$x(\tau_2) \in (\delta, \varepsilon)$	$x(t_1) \geqslant \varepsilon$,
4.*	$x(\tau_2) \geqslant \varepsilon$	$x(t_1) \leqslant x(\tau_2)$,
5.*	$x(\tau_2) \geqslant \varepsilon$	$x(t_1) > x(\tau_2)$.

Consider the first possibility. From lemma 3.9 we have

$$\delta < x(\tau_2) \le x(T_2) < x(t_2). \tag{6.201}$$

For this, equality $x(\tau_2) = x(T_2)$ is possible only when $\tau_0 = T_0$. Further, from lemma 3.9 on interval $0 < x \le x(\tau_2)$, we have

$$y_1^+(x) > y_2^+(x) \text{ and } z_1^+(x) > z_2^+(x). \tag{6.202}$$

From equality $\frac{dz}{dx} = \frac{-cx - b\alpha(x)}{y - f(x)}$ and inequality (6.202) we see that for $0 < x \le x(\tau_2)$ there is fulfilled the inequality

$$\frac{dz_1^+}{dx} > \frac{dz_2^+}{dx}.$$

From this inequality and from inequality (6.200) we obtain

$$z_1^+(x) - z_2^+(x) > \zeta \tag{6.203}$$

for all $x \epsilon [0, x(\tau_2)]$.

According to lemma 3.9 we can assert that inequality

$$x(T_1) < x(t_1) \leq \delta < x(\tau_2) \tag{6.204}$$

takes place on path $\phi(p, t)$. But inequality (6.203) is true for all $x \epsilon [0, x(\tau_2)]$; therefore, we can write

$$z_1^+(x(T_1)) - z_2^+(x(T_1)) > \zeta. \tag{6.205}$$

And from inequalities (6.202) and (6.204), we see that

$$y_1^+(x(T_1)) > y_2^+(x(T_1)). \tag{6.206}$$

But by the definition of the instant of time T_1, we have

$$z_2^+(x(T_1)) - x(T_1) = 0.$$

Therefore, from the definition of function w in equality (5.38) and from inequalities (6.205) and (6.206), we obtain

$$w_1^+(x(T_1)) - w_1^+(x(T_1)) > \frac{1}{2}\zeta^2. \tag{6.207}$$

But on the interval of time $0 \leq t \leq t_1$, as follows from equality (5.39), function w increases along path $\phi(p, t)$ and decreases on the interval of time $T_1 \leq t \leq T_2$. Therefore, from inequality (6.207) we have

$$w_1^+(x(t_1)) > w_2^+(x(T_1)) + \frac{1}{2}\zeta^2, \tag{6.208}$$

$$w_1^+(x(t_1)) > w_2^+(x(t_1)) + \frac{1}{2}\zeta^2. \tag{6.209}$$

In inequality (5.209) expression $w_2^+(x(t_1))$ has sense because of inequality (6.201) and condition 1*. From the definition of the instants of time t_1 and T_1, the definitions of function w, and inequality (6.208), we see that

$$[y_1^+(x(t_1))]^2 - [y_2^+(x(T_1))]^2 > \zeta^2.$$

From lemma 3.9 and inequality (6.39) inequality $y < \nu$ is fulfilled on path $\phi(p, t)$ for $t \epsilon [0, \tau]$. Therefore, we can write

$$y_1^+(x(t_1)) - y_2^+(x(T_1)) > \frac{\zeta^2}{2\nu}. \tag{6.210}$$

Since y decreases with increasing time along path $\phi(p, t)$ for $t \epsilon [T_1, T_2]$, it follows from inequality (6.210) that

$$y_1^+(x(t_1)) - y_2^+(x(t_1)) > \frac{\zeta^2}{2\nu}. \tag{6.211}$$

Now we will prove the truth of inequalities

$$w_1^+(\delta) - w_2^+(\delta) > \frac{1}{2}\zeta^2, \tag{6.212}$$

$$y_1^+(\delta) - y_2^+(\delta) > \frac{\zeta^2}{2\nu}. \tag{6.213}$$

If $x(t_1) = \delta$, these inequalities coincide with inequalities (6.209) and (6.211). But let $x(t_1) < \delta$. From equality (6.32) and inequalities (6.202), true for all $x \epsilon [0, \delta]$, there follows inequality

$$\frac{dw_1^+}{dx} > \frac{dw_2^+}{dx}$$

for $x \epsilon [x(t_1), \delta]$. And from this inequality and inequality (6.209), inequality (6.212) also follows. Analogously, from equality (3.7) and inequality (6.211) we obtain inequality (6.213).

Completely analogously, using lemma 3.9 when $\tau_0 > T_0$, we will show that

$$w_2^+(\delta) > w_3^+(\delta),$$

$$y_2^+(\delta) > y_3^+(\delta).$$

And from these inequalities and inequalities (6.212) and (6.213) we obtain the following estimates:

$$w_1^+(\delta) - w_3^+(\delta) > \frac{1}{2}\zeta^2, \tag{6.214}$$

$$y_1^+(\delta) - y_3^+(\delta) > \frac{\zeta^2}{2\nu}. \tag{6.215}$$

Now we will show that inequality

$$y_1^+(x) > y_3^+(\delta) + \frac{\zeta^2}{4\nu} \tag{6.216}$$

is fulfilled for $x \epsilon [\delta, \epsilon]$. In the same way we will prove that on path $\phi(p, t)$ there is fulfilled inequality

$$x(t_2) > \epsilon, \tag{6.217}$$

and with this inequality, also the definition of symbol $y_1^+(x)$ for $x \epsilon [0, \epsilon]$. Indeed, $y_3^+(\delta) > f(\delta)$ from the definition of the instant of time τ_2. But from condition (6.181), for $\delta \leq x \leq \epsilon$ there takes place the inequality

$$f(x) \leq f(\delta) + c(\epsilon - \delta). \tag{6.218}$$

According to this inequality and (6.184), if inequality (6.216) is fulfilled, then inequality (6.217) is also fulfilled. Thus, inequality (6.216) is proved. This inequality is fulfilled for $x = \delta$, as follows from (6.215); therefore, because of the continuity it is also fulfilled for $x > \delta$, but sufficiently close to δ.

Now suppose that there exists an $x^* \epsilon [\delta, \epsilon]$ such that

$$y_1^+(x^*) = \frac{\zeta^2}{4\nu} + y_3^+(\delta). \tag{6.219}$$

Moreover, suppose that x^* is the first point in which inequality (6.216) is violated, i.e., suppose that inequality (6.216) is fulfilled for $x \epsilon [\delta, x^*)$. Because of equality (3.7), on interval $\delta \leq x \leq x^*$ there is fulfilled the inequality

$$\frac{dy_1^+}{dx} > -\frac{\varepsilon}{y - f(x)}.$$

Using inequalities (6.216) and (6.218) we obtain

$$\frac{dy_1^+}{dx} > -\frac{\varepsilon}{y_3^+(\delta) + \frac{\zeta^2}{4\nu} - f(\delta) - c(\varepsilon - \delta)}.$$

Since $y_3^+(\delta) > f(\delta)$, consequently

$$\frac{dy_1^+}{dx} > -\frac{\varepsilon}{\frac{\zeta^2}{4\nu} - c(\varepsilon - \delta)}.$$

And thus, from (6.184) we obtain

$$\frac{dy_1^+}{dx} > -\frac{5\nu\varepsilon}{\zeta^2}.$$

Integrating the last inequality from $x = \delta$ to $x = x^*$, we have

$$y_1^+(x^*) - y_1^+(\delta) > -\frac{5\nu\varepsilon}{\zeta^2}(\varepsilon - \delta).$$

Therefore, from (6.184) we conclude that

$$y_1^+(x^*) - y_1^+(\delta) > -\frac{\zeta^2}{4\nu}.$$

From this inequality and from inequality (6.215), we obtain

$$y_1^+(x^*) > y_3^+(\delta) + \frac{\zeta^2}{4\nu}.$$

And this contradicts equality (6.219). The contradiction obtained proves that inequality (6.216) is fulfilled for all $x \epsilon [\delta, \epsilon]$.

Now estimate the quantity $w_1^+(\epsilon)$ below. From equality (6.32), for $x \epsilon [\delta, \epsilon]$, we see that

$$\frac{dw_1^+}{dx} > (1-b)\frac{-\varepsilon}{y-f(x)}\alpha(x).$$

Thus, from inequalities (6.181), (6.184), (6.216) and (6.218) there follows inequality

$$\frac{dw_1^+}{dx} > (1-b)\frac{-5\nu\varepsilon}{\zeta^2}\alpha(\delta).$$

Integrating this inequality from $x = \delta$ to $x = \epsilon$, because of (6.184) we obtain

$$w_1^+(\varepsilon) - w_1^+(\delta) > -(1-b)\frac{\zeta^2}{4\nu}\alpha(\delta). \tag{6.220}$$

Below, estimate the quantity $w_1^-(\epsilon)$. For the variation of x along path $\phi(p, t)$ from ϵ to $x(t_2)$ and returning from $x(t_2)$ to ϵ, $\phi(p, t)$ cannot intersect plane $z - x = 0$, since z decreases along $\phi(p, t)$ and $z_1^+(\epsilon) < \epsilon$ conditions 1*. Therefore, function w decreases for such a variation of x on path $\phi(p, t)$. Consequently we have

$$y_1^-(\epsilon) > -\nu, \tag{6.221}$$

since in the opposite case it would result that

$$w_1^-(\varepsilon) \geqslant \frac{1}{2}\nu^2 \geqslant w_1^+(x(t_1)) > w_1^+(\varepsilon). \tag{6.222}$$

But inequality (6.37) takes place for $x \geq \epsilon$. Integrating this inequality from $y_1^+(\epsilon)$ to $y_1^-(\epsilon)$, we obtain

$$w_1^+(\varepsilon) - w_1^-(\varepsilon) < (1-b)\delta^4(y_1^+(\varepsilon) - y_1^-(\varepsilon)).$$

And thus from (6.221) we find that

$$w_1^+(\varepsilon) - w_1^-(\varepsilon) < 2(1-b)\nu\delta^4. \tag{6.223}$$

Estimate the quantity $\frac{1-b}{4}\alpha(\delta)$. From (6.11) we have

$$\frac{1}{4}(1-b)\alpha(\delta) \leqslant \frac{1}{4}(H-bH)\delta.$$

Consequently, from (6.3) and (6.7),

$$\frac{1}{4}(1-b)\alpha(\delta) \leqslant \frac{1}{4}(H+c)\delta < 0.025. \tag{6.224}$$

Accordingly, from (6.220), (6.223) and (6.185) we obtain

$$w_1^+(\delta) - w_1^-(\varepsilon) < \frac{1}{2}\zeta^2.$$

From this inequality and inequality (6.214) we have

$$w_1^-(\epsilon) > w_3^+(\delta).$$

Therefore, from the definition of function w in equality (5.38) we obtain

$$\frac{1}{2}[y_1^-(\varepsilon)]^2 + \frac{1}{2}[z_1^-(\varepsilon)-\varepsilon]^2 > \frac{1}{2}[y_3^+(\delta)]^2 + \frac{1}{2}[z_3^+(\delta)-\delta]^2$$

or

$$\begin{gathered}[y_1^-(\varepsilon)]^2 + [z_1^-(\varepsilon)-\delta]^2 + (\varepsilon-\delta)^2 \\ -2[z_1^-(\varepsilon)-\delta](\varepsilon-\delta) > [y_3^+(\delta)]^2 + [z_3^+(\delta)-\delta]^2.\end{gathered}$$

However, along path $\phi(p, t)$, z decreases with increasing time on interval $[0, \tau]$; consequently, $z_1^-(\epsilon) > z_3^+(\delta)$. Moreover, from condition 1* we have $z_1^+(\delta) \leq \delta$, but $z_1^-(\epsilon) < z_1^+(\delta)$; consequently, $z_1^-(\epsilon) < \delta$. Therefore, from the last inequality we obtain

$$[y_1^-(\varepsilon)]^2 > [y_3^+(\delta)]^2 - (\varepsilon-\delta)^2 - 2\delta(\varepsilon-\delta).$$

Since $y_3^+(\delta) > f(\delta)$ and because of (6.2), (6.3) and (6.11), $f(\delta) > \left(c + \frac{1}{c}\right)\delta$. Then from the last inequality and from inequality (6.6) we find

$$[y_1^-(\varepsilon)]^2 > \delta^2\left[\left(\frac{1}{c}+c\right)^2 - \delta^2 - 2\delta\right],$$

and thus we have

$$|y_1^-(\varepsilon)| > \delta\left[\left(\frac{1}{c}+c\right) - \delta^2 - 1.5\delta\right].$$

Therefore, due to (6.7) we obtain the following inequalities:

$$|y_1^-(\varepsilon)| > c\delta + \frac{0.8\,\delta}{c}, \tag{6.225}$$

$$|y_1^-(\varepsilon)| > 0.8\,c\delta + \frac{\delta}{c}. \tag{6.226}$$

But it is clear that $y_1^-(\epsilon) < f(\epsilon)$; accordingly, from (6.13) we obtain

$$y_1^-(\epsilon) < c\epsilon + \alpha(\epsilon) < c\epsilon + \delta^4. \tag{6.227}$$

We will show that

$$c\varepsilon + \delta^4 < c\delta + \frac{0.8}{c}\delta. \tag{6.228}$$

Due to (6.184) we have $c(\epsilon - \delta) < \delta^2$; therefore, for the proof of inequality (6.228) we need only establish that $\delta^2 + \delta^4 < \frac{0.8}{c}\delta$, and this inequality results from (6.7). From inequalities (6.225), (6.227) and (6.228), we have $y_1^-(\epsilon) < 0$, and from (6.226) we obtain

$$y_1^-(\varepsilon) < -\left(0.8\,c + \frac{1}{c}\right)\delta. \tag{6.229}$$

From this inequality and from equality (6.32) there follows inequality

$$\frac{dw_1^-}{dx} < (1-b)\frac{\varepsilon}{\left(0.8\,c + \frac{1}{c}\right)\delta + c\delta}\,\alpha(x)$$

for $x\epsilon[\delta, \epsilon]$. Integrating this inequality from $x = \delta$ to $x = \epsilon$ and using (6.181), we obtain

$$w_1^-(\varepsilon) - w_1^-(\delta) < \frac{(1-b)\,\varepsilon\,(\varepsilon-\delta)\,\alpha(\delta)}{\left(1.8\,c + \frac{1}{c}\right)\delta}.$$

Accordingly, from (6.184) and (6.224) we find

$$w_1^-(\varepsilon) - w_1^-(\delta) < 0.05\,\frac{\zeta^2}{\nu}. \tag{6.230}$$

From this inequality and inequalities (6.220), (6.223), (6.224) and (6.185), we have

$$w_1^+(\delta) - w_1^-(\delta) < \frac{1}{2}\zeta^2. \tag{6.231}$$

Here, the designation $w_1^-(\delta)$ is defined since, as mentioned above, $z_1^-(\delta) < z_1^-(\epsilon) < \delta$ and, consequently, decreases with decreasing x for $\delta \leq x \leq \epsilon$, $y_1^-(x)$ and path $\phi(p, t)$ in this interval does not intersect surface $y - f(x) = 0$.

From inequalities (6.231) and (6.214) we obtain

$$w_1^-(\delta) - w_3^+(\delta) > 0. \tag{6.232}$$

As mentioned before, path $\phi(p, t)$ intersects plane $z - x = 0$ for the first time after $t = t_2$ for $x < \delta$, i.e., $x(t_4) < \delta$ on $\phi(p, t)$. Moreover, from inequality (6.229) it follows that $y(t_4) < 0$, and then the instants t_3 and t_5 are defined on path $\phi(p, t)$. Because $z_3^+(\delta) < \delta$, it is clear that the expression $z_3^-(\delta)$ is defined and $z_3^-(\delta) < \delta$ since z decreases with increasing time for $t\epsilon[0, \tau]$ on path $\phi(p, t)$. Moreover, it is obvious that w decreases for $t\epsilon[t_3^+(\delta), t_3^-(\delta)]$ along path $\phi(p, t)$ since $z - x < 0$ for such t. Consequently,

$$w_3^-(\delta) < w_3^+(\delta).$$

Thus from (6.232) we obtain

$$w_1^-(\delta) > w_3^-(\delta). \tag{6.233}$$

We will show that

$$w_1^-(x(t_4)) > w_3^-(x(t_4)). \tag{6.234}$$

Indeed, from inequality (6.233) we see that

$$y_1^-(\delta) < y_3^-(\delta), \tag{6.235}$$

since

$$z_1^-(\delta) < z_3^-(\delta), \tag{6.236}$$

and $z_1^-(\delta) < \delta$. From equality (6.32) and inequalities (6.235) and (6.236), for $x \leq \delta$ and sufficiently close to δ there is fulfilled the inequality

$$\frac{dw_1^-}{dx} < \frac{dw_3^-}{dx}. \tag{6.237}$$

We will show that this inequality is fulfilled for $x\epsilon[x(t_4), \delta]$. Assume to the contrary that there exists a $x^*\epsilon[x, (t_4), \delta]$ such that

$$\frac{dw_1^-}{dx} = \frac{dw_3^-}{dx}, \tag{6.238}$$

for $x = x^*$ and inequality (6.237) is fulfilled for $x \epsilon (x^*, \delta]$. From inequalities (6.237) and (6.233) we write

$$w_1^-(x^*) > w_3^-(x^*). \tag{6.239}$$

However, since $z_1^-(x^*) > z_3^-(x^*)$ and $z_1^-(x^*) \leq x^*$, then from (6.239) $y_1^-(x^*) < y_3^-(x^*)$ and for $x = x^*$ there is fulfilled inequality

$$\frac{dw_1^-}{dx} < \frac{dw_3^-}{dx},$$

which contradicts equality (6.238). The contradiction obtained thus proves the fulfillment of inequality (6.237) for all $x \epsilon [x(t_4), \delta]$. From inequalities (6.233) and (6.237), inequality (6.234) results.

We conceive of the following four cases.

(α) Suppose that $z(\phi(p, \tau)) \leq 0$. Then, as is easily seen, w decreases along path $\phi(p, t)$ for $t \epsilon [t_3^-(x(t_4), \tau]$. Consequently, from inequality (6.234) there follows inequality

$$w(\tau) < w(t_4) \tag{6.240}$$

on path $\phi(p, t)$. But on interval $t_4 \leq t \leq t_5$ on path $\phi(p, t)$ inequality $z - x > 0$ is fulfilled; consequently, w increases on this interval along path $\phi(p, t)$, and then

$$w(\phi(p, t_5)) > w(\phi(p, \tau)). \tag{6.241}$$

We will show that

$$w(\phi(p, t_5)) < w(p). \tag{6.242}$$

Since z decreases along $\phi(p, t)$ for $t \epsilon [0, \tau]$, we have

$$z(\phi(p, t_5)) < z(p). \tag{6.243}$$

Moreover, from the definition of instant t_5 we obtain

$$y(\phi(p, t_5)) = 0 \text{ and } z(\phi(p, t_5)) - x(\phi(p, t_5)) > 0.$$

From these relations and the definition of function w in equality (5.38), inequality (6.242) thus follows. Inequality (6.188) follows from inequalities (6.241) and (6.242).

(β) Suppose that $t_5 = T_0$ and $z(\phi(p, \tau)) > 0$. Because of inequality (6.234) and because function w increases for $t \epsilon [t_4, t_5]$ and decreases for $t \epsilon [t_3^-(x(t_4)), \tau_4]$ along path $\phi(p, t)$, there follows inequality

$$w_1^-(x(\tau_4)) > w_3^-(x(\tau_4)). \tag{6.244}$$

We will show that inequality

$$w_1^-(x) > w_3^-(x) \tag{6.245}$$

is fulfilled for all $x \epsilon [0, x(\tau_4)]$. Assume to the contrary that there exists a $x^* \epsilon [0, x(\tau_4)]$ such that inequality (6.245) is fulfilled for $x \epsilon (x^*, x(\tau_4)]$ and

$$w_1^-(x^*) = w_3^-(x^*) \tag{6.246}$$

for $x = x^*$. But for $x \epsilon [0, x(\tau_4)]$ we have $z_1^-(x) > z_3^-(x) \geq x$. Then from inequality (6.246) $y_1^-(x^*) > y_2^-(x^*)$. Thus we obtain

$$\left.\frac{dw_1^-}{dx}\right|_{x=x^*} < \left.\frac{dw_3^-}{dx}\right|_{x=x^*}.$$

But from the continuity, we will have

$$\frac{dw_1^-}{dx} < \frac{dw_3^-}{dx}$$

for $x > x^*$, but sufficiently close to x^*. The last inequality in combination with inequality (6.245) contradicts equality (6.246). This contradiction proves that inequality (6.245) is fulfilled for all $x \epsilon [0, x(\tau_4)]$ and, *in particular, that inequality* (6.241) is fulfilled for $x = 0$. As above, we obtain inequality (6.188) from inequality (6.241).

(γ) Now let $0 < x(t_5) \leq x(\tau_4)$ and $z(\phi(p, \tau)) > 0$. In precisely the same way as in the preceding case, we will prove that inequality (6.245) is fulfilled for $x \epsilon [x(t_5), x(\tau_4)]$, and then we will have

$$w_1^-(x(t_5)) > w_3^-(x(t_5)).$$

Substituting for function w its expressions in x, y, and z, we obtain

$$\begin{gathered}[y(t_5)]^2 + [z(t_5) - x(t_5)]^2 \\ > [y_3^-(x(t_5))]^2 + [z_3^-(x(t_5)) - x(t_5)]^2.\end{gathered}$$

However, by definition of instant t_5, we have $y(t_5) = 0$. Moreover, it is clear that $z(t_5) > z_3^-(x(t_5)) \geq x(t_5)$. Therefore, from the last inequality we obtain

$$\begin{gathered}[y_3^-(x(t_5))]^2 + [z_3^-(x(t_5))]^2 + [x(t_5)]^2 - 2z_3^-(x(t_5))\,x(t_5) \\ < [z(t_5)]^2 + [x(t_5)]^2 - 2z(t_5)\,x(t_5).\end{gathered}$$

Thus, we have

$$[y_3^-(x(t_5))]^2 + [z_3^-(x(t_5))]^2 < [z(t_5)]^2. \tag{6.247}$$

Since $|y|$ and $|z|$ decrease with increasing time along path $\phi(p, t)$ for $t \epsilon [t_3^-(x(t_5)), \tau]$, from inequality (6.247) we obtain inequality

$$[y(\tau)]^2 + [z(\tau)]^2 < [z(t_5)]^2.$$

But $0 < z(t_5) < z(p)$; therefore,

$$[y(\tau)]^2 + [z(\tau)]^2 < [z(p)]^2 \leq 2w(p).$$

This inequality thus proves inequality (6.188).

Go now to the last case.

(δ) Suppose that $z(\phi(p, \tau)) > 0$ and $0 < x(\tau_4) < x(t_5)$ on path $\phi(p, t)$. Since function w increases along with time on path $\phi(p, t)$ on the interval of time $[t_4, t_5]$, from inequality (6.234) there follows the relation

$$w(t_5) > w(\tau_4). \tag{6.248}$$

Rewrite this inequality in the following form:

$$y^2(t_5) + [z(t_5) - x(t_5)]^2 > y^2(\tau_4) + [z(\tau_4) - x(\tau_4)]^2.$$

Since we have $y(t_5) = 0$ from the definition of instant t_5, we can write

$$z^2(t_5) + x^2(t_5) - 2x(t_5)z(t_5) > y^2(\tau_4) + z^2(\tau_4) + x^2(\tau_4) - 2x(\tau_4)z(\tau_4).$$

But from the definition of τ_4 we have $x(\tau_4) = z(\tau_4)$; consequently, we can write

$$z^2(t_5) + x^2(t_5) - 2x(t_5)z(t_5) > y^2(\tau_4) + z^2(\tau_4) - x^2(\tau_4).$$

Since $x(\tau_4) < x(t_5)$ by hypothesis, from the last inequality we obtain

$$z^2(t_5) - 2x(t_5)z(t_5) + 2x^2(t_5) > y^2(\tau_4) + z^2(\tau_4).$$

However, $z \geq x$ for $t \epsilon [t_4, t_5]$ on path $\phi(p, t)$; therefore, we have

$$y^2(\tau_4) + z^2(\tau_4) < z^2(t_5). \tag{6.249}$$

Since $|y|$ and $|z|$ decrease with increasing time along $\phi(p, t)$ for $t \epsilon [\tau_4, \tau]$, from the last inequality, as in case (γ), we obtain inequality (6.188).

Thus, in case 1* inequality (6.182) is fulfilled.

Consider now cases 2* and 3*. In these cases, as in the proof of inequality (6.208), we establish the inequality

$$w(t_1) > w(\tau_1) + \frac{1}{2}\zeta^2. \tag{6.250}$$

Since function w on interval $\tau_0 \leq t \leq \tau_2$ of path $\phi(p, t)$ has only one maximum which is for $t = \tau_1$, from (6.250) it follows that

$$w(t_1) > w_3^+(\delta) + \frac{1}{2}\zeta^2. \tag{6.251}$$

If $x(t_4) \leq \epsilon$, then, as in case 1*, from inequalities (6.250) and (6.251) we obtain the inequalities

$$y(t_4) < 0 \tag{6.252}$$

and

$$w(t_4) > w(\tau_4). \tag{6.253}$$

But if $x(t_4) > \epsilon$, inequality (6.253) obviously preserves its truth.

We will show that inequality (6.252) is also fulfilled in this case. Indeed, assume to the contrary that $y(t_4) \geq 0$. But t_4 is the instant of intersection of path $\phi(p, t)$ with plane $z - x = 0$. For $t < t_4$ and sufficiently close to t_4, path $\phi(p, t)$ lies in domain $\{x > 0,\ z - x < 0\}$. Therefore, as proven above, $y(t_4)$ must satisfy inequality (6.49). But by hypothesis $x(t_4) > \epsilon$; therefore, from inequality (6.49), lemma 3.9 and condition (6.13), $y(t_4) < (1 - b)\delta^4$. Since $z - x = 0$ for $t = t_4$ on $\phi(p, t)$, then

$$w(t_4) = \frac{1}{2}y^2(t_4) < \frac{1}{2}(1 - b)^2\delta^8. \tag{6.254}$$

On the other hand, from (6.253) we have

$$\begin{aligned} w(t_4) > w(\tau_1) \geqslant w_3^+(\delta) \geqslant \frac{1}{2}[y_3^+(\delta)]^2 \\ \geqslant \frac{1}{2}f^2(\delta) \geqslant \frac{1}{2}\left(c + \frac{1}{c}\right)^2\delta^2 \end{aligned}$$

Since we are considering only case $c^2 + b > 0$, the last inequality contradicts inequality (6.254). Consequently, in the cases considered, inequalities (6.252) and (6.253) are fulfilled. From these inequalities, as in case 1*, it is easy to obtain inequality (6.188).

In cases 4* and 5*, as in the preceding cases, we establish inequalities (6.250)–(6.253). And from these inequalities it is not difficult to verify inequality (6.188).

In case II_2, from lemma 3.9 we have

$$y_2^+(x(\tau_2)) > y(\tau_2). \tag{6.255}$$

Moreover, since z decreases along path $\phi(p, t)$ for $t\epsilon[0, \tau]$, we have

$$z_2^+(x(\tau_2)) > z(\tau_2). \tag{6.256}$$

From these inequalities and the definition of function w, we find

$$w_2^+(x(\tau_2)) > w(\tau_2). \tag{6.257}$$

As in case II_1, from lemma 3.9 it is easy to obtain the inequality

$$w_2^+(x(\tau_2)) + \frac{1}{2}\zeta^2 < w_1^+(x(\tau_2)). \tag{6.258}$$

From these inequalities, as in case II_1, it is not difficult to obtain inequality (6.188).

Now consider case I and introduce the following designations. We assume $\tau' = \tau_1$ if relation (6.191) is true and $\tau' = \tau_2$ if relation (6.192) is true. Thus, $z(\tau') - x(\tau') = 0$ on path $\phi(p, t)$. Since, z decreases for $t\epsilon[0, \tau]$ along $\phi(p, t)$, then for $t\epsilon[0, t_2]$ we have

$$z(t) > z(\tau'). \tag{6.259}$$

Because $z(t_1) = x(t_1)$ from the definition of instant t_1, from (6.259) we obtain

$$x(t_1) > z(\tau') = x(\tau'). \tag{6.260}$$

Then from the continuity of function x(t) on path $\phi(p, t)$, we can assert that there exists a $t'\epsilon(0, t_1)$ such that

$$x(t') = x(\tau'). \tag{6.261}$$

If relation (6.191) is fulfilled, function x(t) on path $\phi(p, t)$ for $t\epsilon[\tau', \tau]$ has only one maximum which is for $t = \tau_2$; but if relation (6.192) is fulfilled, then function x(t) decreases along $\phi(p, t)$ for $t\epsilon[\tau', \tau]$. Thus from the conditions of case I, inequality

$$x(t) \leq x(\tau_2) \leq \delta. \tag{6.262}$$

is fulfilled on $\phi(p, t)$ for $t\epsilon[\tau', \tau]$. From equality (6.261) and lemma 3.9, on path $\phi(p, t)$ there is true the inequality

$$y(t') > y(\tau'), \; z(t') > z(\tau'). \tag{6.263}$$

Consider function v as defined by equality (4.98). We will show that on path $\phi(p, t)$ there is fulfilled the inequality

$$v(t') > v(\tau'). \tag{6.264}$$

From equalities (4.98) and (6.261) and inequality (6.263), it is sufficient to show that for this

$$\frac{1}{2} y^2(t') - (1-b)\, y(t')\, \alpha(x(t'))$$
$$> \frac{1}{2} y^2(\tau') - (1-b)\, y(\tau')\, \alpha(x(\tau')). \tag{6.265}$$

Following from conditions (6.3) and (6.11),

$$(1 - b)\alpha(x(t')) < \alpha(x(t')) + cx(t') = f(x(t')). \tag{6.266}$$

But $t' \epsilon [0, t_1]$; therefore,

$$y(t') \geq f(x(t')). \tag{6.267}$$

Thus, from (6.266) there follows inequality

$$y(t') > (1 - b)\alpha(x(t')). \tag{6.268}$$

If $\tau' = \tau_1$, in an analogous manner we obtain

$$y(\tau') \geq (1 - b)\alpha(x(\tau')). \tag{6.269}$$

If $\tau' = \tau_2$ (i.e., if relation (6.192) is realized), then $z(\tau') = x(\tau')$ and path $\phi(p, t)$ lies in domain $\{z - x < 0\}$ for $t > \tau'$, but sufficiently close to τ'. Thus, from inequality (6.49) inequality (6.269) is fulfilled in this case. From inequalities (6.268), (6.269) and equality (6.261), inequality (6.265) follows and from it, (6.264) also.

Since x increases monotonically along $\phi(p, t)$ for $t \epsilon [0, t_1]$, for $t \epsilon [0, t']$ we have

$$x \leq x(t') = x(\tau') \leq \delta. \tag{6.270}$$

From inequality (6.270), condition (6.180) and equality (4.99), the following inequality results:

$$v(p) > v(\phi(p, t')). \tag{6.271}$$

From inequality (6.262) condition (6.180) and equality (4.99), we obtain

$$v(\phi(p, t')) > v(\phi(p, \tau)). \tag{6.272}$$

From inequalities (6.264), (6.271) and (6.272), there follows inequality

$$v(p) > v(\phi(p, \tau)).$$

Since function v and w coincide for $x = 0$, (6.188) follows from the last inequality. Thus, inequality (6.188) is established.

Inequality (6.188) permits the following assertion to be proved.

Theorem 6.2

Suppose that conditions $a > 0$ and $b < 1$ are fulfilled; in addition, suppose that $c^2 + b > 0$. Let function $\alpha(x)$ satisfy conditions (6.11)–(6.13), (6.180) and (6.181), in which numbers h, H, δ and ϵ satisfy inequalities (6.2), (6.3), (6.6)–(6.10), (6.184) and (6.185). Moreover, let equality (6.169) be true for $|x| \leq \frac{\nu}{c}$. Then system (2.15) has periodic motions.

Proof

Let P be trapezoid $p_0p_1q_1q_0p_0$, defined for the proof of theorem 6.1. We will move through point $p \epsilon P$ of trajectory $\phi(p, t)$ of system (2.15). Let $T_p^{(0)}$ be such that $x(T_p^{(0)}) = 0$ and $x > 0$ for $t \epsilon (0, T_p^{(0)})$ on path $\phi(p, t)$. Of course, it is possible that such a finite $T_p^{(0)}$ does not exist; i.e., it can happen that path $\phi(p, t)$ lies in half-space $\{x > 0\}$ for all $t > 0$; then from theorem 3.1 path $\phi(p, t)$ goes to the origin for $t \to +\infty$. In this case, we will consider $T_p^{(0)} = +\infty$ and $\phi(p, T_p^{(0)}) = (0, 0, 0)$. If path $\phi(p, t)$ touches plane $x = 0$ for $t = T_p^{(0)}$ (i.e., if $y(T_p^{(0)}) = 0$ and $z(T_p^{(0)}) > 0$ on the plane), then we will designate by τ_p the first instant after $t = 0$ of the intersection of path $\phi(p, t)$ with plane $x = 0$. That is, we will consider that on $\phi(p, t), x(\tau_p) = 0$, $y(\tau_p) < 0$ and $x(t) \geq 0$ for $t \epsilon [0, \tau_p]$. It can be said that such a τ_p does not exist (i.e., that path $\phi(p, t)$ lies in half-space $\{x \geq 0\}$ for $t \geq 0$), then by theorem 3.1, $\phi(p, t)$ goes to the origin for $t \to +\infty$; in this case we will say that $\tau_p = +\infty$ and $\phi(p, \tau_p) = (0, 0, 0)$. If on interval $0 < t < \tau_p$, path $\phi(p, t)$ touches plane $x = 0$ not only for $t = T_p^{(0)}$, we will designate by $T_p^{(1)}$, $T_p^{(2)}$, ... the instants of contact of path $\phi(p, t)$ with plane $x = 0$ for $t \epsilon (0, \tau_p)$.

As for the proof of theorem 6.1 we introduce the following two functions of the points of trapezoid P:

$$\Delta r(p) = r(\varphi(p, T_p^{(0)})) - r(p), \tag{6.273}$$

$$\Delta\theta(p) = \theta(\varphi(p, 'T_p^{(0)})) - \theta(p) - \pi, \tag{6.274}$$

where r(p) and $\theta(p)$ are the radius vector and the polar angle of point p. Let E be the set of points of trapezoid P on which inequality $\Delta r(p) \geq 0$ is fulfilled and F the set of points of trapezoid P on which $\Delta r < 0$. Both of these sets are nonempty according to lemmas 6.4 and 6.5. On set E, both functions $\Delta r(p)$ and $\Delta\theta(p)$ are continuous. Indeed, let $p \epsilon E$; then $y(\phi(p, T_p^{(0)})) < 0$ since in the contrary case the relation $w(\phi(p, T_p^{(0)})) = \frac{1}{2} z^2(\phi(p, T_p^{( )})) < \frac{1}{2} w(p)$ would hold, and this inequality is not possible since $p \epsilon E$. From inequality $y\phi(p, T_p^{(0)})) < 0$ and from the theorem on integral continuity, functions Δr and $\Delta\theta$ are continuous on set E.

We will now show that set F is open in P. Let $p \epsilon F$. If $y(\phi(p, T_p^{(0)})) < 0$, then from the theorem on integral continuity, along with p in F a certain neighborhood of this point is contained. But let $y(\phi(p, T_p^{(0)})) = 0$. Then following from inequality (6.188),

$$w(\varphi(p, \tau_p)) < w(p). \tag{6.275}$$

Moreover, since $y(\phi(p, T_p^{(k)})) = 0$ and $z(\phi(p, T_p^{(k)})) > 0$, from the decrease of z along path $\phi(p, t)$ for $t \epsilon [0, \tau_p]$, we have

$$w(\phi(p, T_p^{(k)})) < w(p). \tag{6.276}$$

But from the theorem on the continual dependence of the solutions of the initial conditions, on path $\phi(q, t)$ of system (2.15), the initial point $q \epsilon P$ of which lies sufficiently close to p, the points $\phi(q, T_q^{(o)})$ lie arbitrarily close to one of points $\phi(p, T_p^{(k)})$ or $\phi(p, \tau_p)$. Thus, from inequalities (6.275) and (6.276), set F is open in P.

Since set F is open in P and since function Δr is continuous on set E, from lemmas 6.4 and 6.5 there exists a continuum K such that $\Delta r(p) = 0$ if $p \epsilon K$ and K has points both on side $p_0 q_0$ and on side $p_1 q_1$ of trapezoid $p_0 p_1 q_1 q_0 p_0$. But $K \subset E$; consequently, function $\Delta\theta$ is continuous on K. From lemma 6.6, it follows that

$$\Delta\theta(p) > 0 \tag{6.277}$$

if p lies at the intersection of K with side $p_1 q_1$ of trapezoid P. But if point p lies on the intersection of K with segment $p_0 q_0$ of axis Oz, as proved above, $y(\phi(p, \tilde{T}_p^{(o)})) < 0$ and, consequently,

$$\Delta\theta(p) < 0. \tag{6.278}$$

From the continuity of function $\Delta\theta$ on continuum K and from inequalities (6.277) and (6.278), on K there exists a point p_0 such that

$$\Delta\theta(p_0) = 0. \tag{6.279}$$

But from the definition of K we have

$$\Delta r(p_0) = 0. \tag{6.280}$$

From equalities (6.279) and (6.280), $\phi(p_0, t)$ is a path of periodic motion of system (2.15).

The theorem is proved.

Corollary

Suppose that $a > 0$, $b < 1$ and $c^2 + b > 0$. Suppose further that function $\alpha(x)$ is differentiable for $0 \leq x \leq K\delta$ and there are fulfilled the following inequalities:

$$hx \leqslant \alpha(x) \leqslant Hx, \ \frac{d\alpha}{dx} > 0 \quad \text{for} \quad 0 \leqslant x < K\delta, \tag{6.281}$$

$$0 < \alpha(x) < \alpha(K\delta) \quad \text{for} \quad K\delta < x \leqslant K\varepsilon, \tag{6.282}$$

$$0 < \alpha(x) < K\delta^4 \quad \text{for} \quad K\varepsilon \leqslant x \leqslant \frac{K\nu}{c}, \tag{6.283}$$

where numbers h, H, δ and ϵ satisfy inequalities (6.2), (6.3), (6.6)–(6.10), (6.184) and (6.185), and K is an arbitrary positive number. Suppose moreover that equality (6.169) is true for $|x| \leq \frac{K\nu}{c}$. Then system (2.15) has a periodic motion. For the proof of this corollary, change values $x = Kx_2$, $y = Ky_2$, $z = Kz_2$ and use theorem 6.2.

We now note the following circumstance.

Remark

In theorem 6.2, number h is an arbitrary number greater than $1/c$; therefore, the range of variation of function $\alpha(x)$ given in theorem 4.5 cannot be broadened. That is, whatever the value of $\chi > \frac{1}{c}$, a function $\alpha(x)$ can be found satisfying the inequality

$$0 < \alpha(x)x \leq \chi x^2 \text{ for } x \neq 0, \tag{6.284}$$

such that system (2.15) will have periodic motions and, consequently, its null solution will not be stable in the whole.

In conclusion, we formulate the following theorem resulting from theorems 4.1–4.4 and 6.2.

Theorem 6.3

In order for the null solution of system (2.15) to be globally stable for any function f(x) satisfying the generalized Hurwitz conditions, it is necessary and sufficient that the conditions of one of the following three cases be fulfilled:

$$\begin{aligned}
&\text{I. } a < 0, \ b > 0, \\
&\text{II. } a = 0, \ 0 < b < 1, \\
&\text{III. } a > 0, \ b < 0, \ \frac{a^2}{(1-b)^2} + b \leqslant 0.
\end{aligned}$$

Chapter VII. ON THE INSTABILITY OF MOTION AND PERIODIC SOLUTIONS. GENERAL CASES

Here we consider system (2.8) for condition $d \neq 0$. For this we direct our attention to those cases in chapter IV for which there was no success in establishing the global stability of the null solution for any nonlinear function $f(x)$ satisfying the generalized Hurwitz conditions (2.9)–(2.12). Thus we will consider cases 8, 11, 12, 13, 14, 16, 17, 21, and 22. As earlier, let numbers A and B be designated by formula (4.101) and number k by the formula

$$k^2 = c + dA. \tag{7.1}$$

For this, if cases 14, 21 or 22 are fulfilled, we will suppose that inequality

$$A^2 - dA + b > 0 \tag{7.2}$$

is true, and if the conditions of case 16 are fulfilled, we will say that there is fulfilled the relation

$$Ab - Ak^2 + dk^2 < 0. \tag{7.3}$$

For the conditions formulated above imposed on parameters a, b, c and d of system (2.8), we will find conditions (imposed on nonlinearity $f(x)$) such that for their fulfillment not all the solutions of system (2.8) will go to origin for $t \to +\infty$. At the same time we will prove that in the cases considered the Ayzerman problem is answered in the negative. And since we proved the global stability for any $f(x)$ for all the remaining cases of system (2.8), we also obtain the necessary and sufficient conditions in order for the null solution of system (2.8) to be globally stable for any function $f(x)$ satisfying the GHC.

Moreover, in this chapter we will find certain sufficient conditions allowing periodic solutions different from the equilibrium position for system (2.8).

Section 20

In this and the following section, we will exclude case 17 from consideration since it is discussed in section 22 of this chapter.

Following directly from a study of the conditions of the cases considered, $A > 0$ for all these cases. From inequality (2.43) we have $k^2 > 0$ for all cases studied except case 13. Moreover, $k^2 > 0$ in case 13 also for the following reasons: inequality (2.42); $d < 0$ in case 13; in this case number A is the boundary on the left of function $f(x)/x$ for the fulfillment of the GHC.

As earlier, introduce instead of $f(x)$ a new nonlinearity $\gamma(x)$ by the formula

$$\gamma(x) = f(x) - Ax. \tag{7.4}$$

In this and the following sections, analogous to that given in the preceding chapter, we will say that function $\gamma(x)$ obeys certain special conditions approximately the same as those in chapter VI. Suppose that function $\gamma(x)$ satisfies the following conditions:

$$hx^2 \le \gamma(x)x \le Hx^2 \text{ for } 0 \le |x| \le \delta, \tag{7.5}$$

$$0 < \gamma(x)x \le Hx^2 \text{ for } \delta \le |x| \le \epsilon, \tag{7.6}$$

$$0 < \gamma(x) \operatorname{sign} x \le \delta^4 \text{ for } |x| \ge \epsilon. \tag{7.7}$$

We will designate these conditions as conditions $E(h, H, \delta)$ or simply as conditions E. In conditions E, the quantity ϵ obeys the inequality

$$0 < \epsilon - \delta \le \delta^2, \tag{7.8}$$

and the numbers h and H obey the inequality

$$H \geqslant h > \frac{k^2(dA + k^2 - b)}{Ak^2 - dk^2 - Ab}. \tag{7.9}$$

As in the preceding chapter, we will say that the number δ is sufficiently small. However, for a simplified discourse we will not evaluate quantity δ since this was given in chapter VI in inequalities (6.7)–(6.10), and we suppose only that this quantity is sufficiently small. We will understand this in the sense that a $\delta_0 > 0$ can be found for fixed h and H, such that the asserted relations will be fulfilled for $\delta \le \delta_0$.

In the following we will frequently use the notation $O(\delta^r)$. This symbol, as usual, will designate the quantity for which constants N and δ_0 can be found such that $|O(\delta^r)| \le N\delta^r$ for $\delta \le \delta_0$.

System (2.8) is rewritten in the notations of (7.4) in the form

$$\left.\begin{aligned} \frac{dx}{dt} = y - Ax - \gamma(x), \quad \frac{dy}{dt} = z - cx - dAx - d\gamma(x), \\ \frac{dz}{dt} = -ax - bAx - b\gamma(x). \end{aligned}\right\} \tag{7.10}$$

As before, in system (7.10) we will substitute the change of variables

$$x_1 = A^2x - Ay + z, \quad y_1 = z - k^2x, \quad z_1 = ky, \tag{7.11}$$

$$x=\frac{x_1-y_1+\frac{A}{k}z_1}{A^2+k^2},\quad y=\frac{z_1}{k},\quad z=\frac{k^2x_1+A^2y_1+Akz_1}{A^2+k^2}. \tag{7.12}$$

Then we will have

$$\begin{gathered}\frac{dx_1}{dt}=-Ax_1-(A^2-dA+b)\gamma(x)\\ \frac{dy_1}{dt}=-kz_1+(k^2-b)\gamma(x),\ \frac{dz_1}{dt}=ky_1-dk\gamma(x).\end{gathered} \tag{7.13}$$

In the following we will frequently compare the solutions of system (2.8) with the solutions of that system which is obtained if we assume in (2.8) that $\gamma(x) = f(x) - Ax = 0$, i.e., with the solutions of the system

$$\frac{dx}{dt}=y-Ax,\ \frac{dy}{dt}=z-cx-dAx,\ \frac{dz}{dt}=-ax-bAx. \tag{7.14}$$

In the variables x_1, y_1, z_1, this system has the following form:

$$\frac{dx_1}{dt}=-Ax_1,\ \ \frac{dy_1}{dt}=-kz_1,\ \frac{dz_1}{dt}=ky_1. \tag{7.15}$$

In the following, $\psi(p, t)$ will designate those paths of system (7.14) which go through point p of the phase space for $t = 0$.

It is easy to see that the general solution of system (7.15) has the form

$$\left.\begin{gathered}x_1=x_{10}e^{-At},\ \ y_1=y_{10}\cos kt-z_{10}\sin kt,\\ z_1=y_{10}\sin kt+z_{10}\cos kt,\end{gathered}\right\} \tag{7.16}$$

where x_{10}, y_{10} and z_{10} are the values of functions x_1, y_1, z_1 for $t = 0$.

Let p and q be two points of the phase space and designate the distance between them by $\|p - q\|$. Let $\phi(p, t)$ and $\psi(q, t)$ be paths of systems (7.10) and (7.14) respectively. We will consider this path on the interval of time $0 \le t \le T$ where T is some fixed number. Then, it is obvious that

$$\|\varphi(p,t)-\psi(q,t)\|=O(\delta)\cdot t+O(\|q-p\|). \tag{7.17}$$

Estimate (7.17) because the special form of condition $E(h, H, \delta)$ can be improved.

Lemma 7.1

Suppose that function $\gamma(x)$ satisfies condition $E(h, H, \delta)$. Suppose further that points p and q lie in domain $\left\{x=0,\ |x_1|\leqslant\delta,\ y=\frac{z_1}{k}\geqslant\frac{1}{2}\right\}$ and $\|p - q\| = O(\delta^2)$. Then for sufficiently small δ the relation

$$\|\phi(p, t) - \psi(q, t)\| = O(\delta^2) \tag{7.18}$$

is true for $t \epsilon [0, 5\pi/2k]$.

Proof

Directly from formula (7.16) we see that path $\psi(q, t)$ of system (7.14) on the interval of time $0 \leq t \leq \frac{5\pi}{2k}$ intersects plane $x = 0$ three times: for $t = 0$, $t = t_1'$ and $t = t_2'$. In this case it is easy to see that $t_1' = \frac{\pi}{k} + O(\delta)$ and $t_2' = \frac{2\pi}{k} + O(\delta)$. But from estimate (7.17) and the reasoning of section 3 of chapter III, for sufficiently small δ, path $\phi(p, t)$ also intersects plane $x = 0$ three times on the interval of time $[0, 5\pi/2k]$: for $t = 0$, $t = t_1$ and $t = t_2$. For this it is clear that $t_1 = \frac{\pi}{k} + O(\delta)$ and $t_2 = \frac{2\pi}{k} + O(\delta)$.

Furthermore, from formula (7.16) on the interval of time $[0, t_1']$ path $\psi(q, t)$ intersects plane $x = \epsilon$ two times, for $t = \tau_1'$ and $t = \tau_2'$, and also plane $x = 2\epsilon$ two times, for $t = \theta_1$ and $t = \theta_2$. It is clear that $\tau_1' < \theta_1 < \theta_2 < \tau_2'$.

From estimate (7.17) it is not difficult to conclude that path $\phi(p, t)$ of system (7.10) also intersects plane $x = \epsilon$ two times on interval $[0, t_1]$, for $t = \tau_1$ and $t = \tau_2$. Further, from formula (7.16) it follows that $\theta_1 = O(\delta)$. However, from (7.17), relation (7.18) is fulfilled for $t \epsilon [0, \theta_1]$. According to this estimate, $\tau_1 \epsilon (0, \theta_1)$; consequently, point $\phi(p, \theta_1)$ lies in domain $\{x > \epsilon\}$. From condition $E(h, H, \delta)$ and estimate (7.18), relation (7.18) is fulfilled from that time when the points of both paths $\phi(p, t)$ and $\psi(q, t)$ lie in domain $\{x > \epsilon\}$. But from this relation it is clear that both of these paths lie in half-space $\{x > \epsilon\}$ on the interval of time $\theta_1 \leq t \leq \theta_2$. Now designating by θ_3 the instant following θ_2 of the intersection of path $\psi(q, t)$ with plane $x = -2\epsilon$, as above, we will show that

$$\theta_3 - \theta_2 = O(\delta).$$

And thus it follows that relation (7.18) is also fulfilled for $0 \leq t \leq \theta_3$.

Continuing further with the same reasoning, we will prove the lemma. From lemma 7.1 the following relations are obtained:

$$\|\phi(p, t_1) - \psi(q, t_1')\| = O(\delta^2), \tag{7.19}$$

$$\|\phi(p, t_2) - \psi(q, t_2')\| = O(\delta^2). \tag{7.20}$$

Lemma 7.2

Suppose that function $\gamma(x)$ satisfies condition $E(h, H, \delta)$ with a sufficiently small δ. Suppose further that point p lies in domain $\left\{x = 0, |x_1| < \delta^{9/8}, y = \frac{z_1}{k} \geqslant \frac{1}{2}\right\}$; then relation

$$|x_1(\varphi(p, t_2))| < \delta^{3/2} \tag{7.21}$$

is true where, as earlier, t_2 is the first instant after $t = 0$ of the intersection of path $\phi(p, t)$ with half-plane $\{x = 0, y > 0\}$.

Proof

Directly from formula (7.16) and estimate (7.17), for sufficiently small δ, path $\phi(p, t)$ intersects half-plane $\{x = 0, y > 0\}$ for $t = t_2 \epsilon (0, 5\pi/2k)$. Now, as earlier, let t_2' be the first instant after $t = 0$ of the intersection of path $\psi(p, t)$ of system (7.14) with half-plane $\{x = 0, y > 0\}$. Then from the first of formulas (7.16), there exists a positive constant $r < 1$ such that

$$|x_1(\psi(p, t_2'))| < r\delta^{3/2}.$$

Therefore, the assertion of the lemma also follows from (7.20).

In the cases considered, introduce the function of the coordinates of the phase space:

$$w = \frac{1}{2}(z - k^2x)^2 + \frac{1}{2}k^2y^2. \tag{7.22}$$

The time derivative of this function, taken because of the differential equations of system (7.10) as is easily verified, equals

$$\dot{w} = [(k^2 - b)(z - k^2x) - dk^2y]\,\gamma(x). \tag{7.23}$$

In plane $x = 0$, consider the following domain:

$$P = \left\{x = 0,\ |x_1| < \delta^{3/2},\ w > \frac{1}{2}(A^2 + k^2),\ y > 0\right\}.$$

Designate its closure by $\bar{P}$. Take an arbitrary point $p \epsilon \bar{P}$ and consider a path $\phi(p, t)$ of system (2.8). If condition E is fulfilled, then following from the preceding reasoning, there exist instants of time $0 < t_1 < t_2$ such that points $\phi(p, t_1)$ and $\phi(p, t_2)$ lie in plane $x = 0$; on interval $0 < t < t_1$ and $t_1 < t < t_2$, x preserves sign on path $\phi(p, t)$. Accordingly, assign point $p \epsilon \bar{P}$ to $\phi(p, t_2)$ lying in half-plane $\{x = 0, y > 0\}$. Designate by I the transformation thus obtained of the plane of domain $\bar{P}$ into half-plane $\{x = 0, y > 0\}$. From the uniqueness theorem and the theorem on integral continuity, the transformation I is mutually single valued and mutually continuous.

Lemma 7.3

Suppose that parameters a, b, c and d of system (2.8) satisfy one of the three following conditions:

I. The conditions of one of cases 8, 11, 12 or 13.

II. The conditions of one of cases 14, 21 or 22 and inequality

$$A^2 - Ad + b > 0. \tag{7.24}$$

III. The conditions of case 16 and inequality

$$Ab - Ak^2 + dk^2 < 0. \tag{7.25}$$

Suppose further than function $\gamma(x) = f(x) - Ax$ satisfies condition $E(h, H, \delta)$ with sufficiently small δ. Then there is true relation

$$I(\overline{P}) \subset P. \tag{7.26}$$

Proof

From relation (4.153) we see that the inequality

$$b - k^2 - dA < 0 \tag{7.27}$$

is fulfilled in the conditions of the lemma. We will show, moreover, that inequality (7.25) is always fulfilled in the conditions of the lemma. If condition I of our lemma is fulfilled, from (7.27) and $d < 0$, $b - k^2 < 0$, and thus (7.25) also follows.

Suppose that condition II of the lemma is fulfilled. If case 14 occurs, inequality (4.25) follows immediately because $d < 0$ and $b < 0$ in this case. But if case 21 or 22 is realized, then we have $A - d > 0$ from (7.24) and inequality (7.25) from $k^2 > 0$ and $b \leq 0$.

Finally, if condition III is fulfilled, inequality (7.25) is immediately given.

According to lemmas 7.1 and 7.2, for the proof of relation (7.26) it is sufficient to prove only the inequality

$$w(p) < w(\phi(p, t_2)), \tag{7.28}$$

if

$$p \in \left\{x = 0,\ |x_1| \leqslant \delta^{3/4},\ \frac{1}{2}(A^2 + k^2) \leqslant w \leqslant 2(A^2 + k^2),\ y > 0\right\}.$$

First we will prove that inequality

$$w(p) < w(\phi(p, t_1)) \tag{7.29}$$

is fulfilled for

$$p \in \left\{x = 0, \ |x_1| \leqslant \delta^{3/2}, \ \frac{1}{2}(A^2 + k^2) \leqslant w \leqslant 2(A^2 + k^2), \ y > 0\right\}$$

As usual, in inequalities (7.28) and (7.29) t_1 and t_2 are sequences of the instants of the intersection of path $\phi(p, t)$ with plane $x = 0$.

As mentioned before, path $\phi(p, t)$ intersects plane $x = \epsilon$ twice on the interval of time $[0, t_1]$, for $t = \tau_1$ and $t = \tau_2$. It is further clear that $\phi(p, t)$ also intersects plane $x = \delta$ twice on interval $0 \leq t \leq t_1$. Let $t = T_1$ and $t = T_2$ be a sequence of the intersection of path $\phi(p, t)$ with plane $x = \delta$. Then obviously, $0 < T_1 < \tau_1 < \tau_2 < T_2 < t_1$.

Calculate the increase of function w along path $\phi(p, t)$ for a variation of time from $t = 0$ to $t = t_1$. Let y_0 and z_0 be the y component and the z component of point p *and* $y^{(1)}$ *and* $z^{(1)}$ *be the corresponding* coordinates of point $\phi(p, t_1)$. From the same definition of point p we have

$$z_0 = Ay_0 + O(\delta^{3/2}). \tag{7.30}$$

From lemmas 7.1 and 7.2 and formula (7.16) it is easy to see that

$$y^{(1)} = -y_0 + O(\delta^{3/2}), \quad z^{(1)} = -z_0 + O(\delta^{3/2}). \tag{7.31}$$

Evaluate the quantities y and z on path $\phi(p, t)$ for $t \epsilon [0, \tau_1]$ from the small to the large sequence relative to δ. We have

$$\frac{dy}{dx} = \frac{z - cx - df(x)}{y - f(x)}. \tag{7.32}$$

Since function $\gamma(x) = f(x) - Ax$ satisfies condition $E(h, H, \delta)$, from the last equality and equality (7.30) we obtain the following relation which is true on path $\phi(p, t)$ for $t \epsilon [0, \tau_1]$:

$$\frac{dy}{dx} = A + O(\delta). \tag{7.33}$$

Integrating this equality along path $\phi(p, t)$ from $t = 0$ to $t = \tau_1$ we obtain

$$y = y_0 + Ax + O(\delta^2). \tag{7.34}$$

As from equality

$$\frac{dz}{dx} = \frac{-ax - bf(x)}{y - f(x)} \tag{7.35}$$

we see that the evaluation

$$\frac{dz}{dx} = O(\delta) \tag{7.36}$$

is true on path $\phi(p, t)$ for $t \epsilon [0, \tau_1]$. Integrating this relation along path $\phi(p, t)$ for $t \epsilon [0, \tau_1]$, we obtain

$$z = Ay_0 + O(\delta^{3/2}) \tag{7.37}$$

because of (7.30).

Now evaluate quantities y and z on $\phi(p, t)$ for $t \epsilon [\tau_2, t_1]$ in the same way. From equality (7.32), because of equalities (7.30) and (7.31) as before, we obtain (7.33). And from this equality and (7.31), we conclude that on path $\phi(p, t)$ for $t \epsilon [\tau_2, t_1]$ there is fulfilled the evaluation

$$y = -y_0 + Ax + O(\delta^{3/2}). \tag{7.38}$$

Analogously, from equality (7.36) we obtain (7.37) and from it,

$$z = -Ay_0 + O(\delta^{3/2}) \tag{7.39}$$

on $\phi(p, t)$ for $t \epsilon [\tau_2, t_1]$.

Evaluate the increase of function w along path $\phi(p, t)$ for a passing through of the last strip $0 \leq x \leq \epsilon$. For this we will consider function w on path $\phi(p, t)$ for $t \epsilon [0, \tau_1]$ and $t \epsilon [\tau_2, t_1]$ as functions of the x component. For this, multiple values immediately arise since path $\phi(p, t)$ passes through strip $0 \leq x \leq \epsilon$ twice on the interval of time $[0, t_1]$. To avoid this multiple valuedness, we will assign to function w the sign + on time interval $[0, \tau_1]$ and the sign – on interval $[\tau_2, t_1]$. Accordingly this will restore the single valuedness.

Dividing equality (7.23) by the first equation of system (7.10), we obtain

$$\frac{dw}{dx} = \frac{(k^2 - b) z - dk^2 y - k^2 (k^2 - b) x}{y - Ax - \gamma(x)} \gamma(x). \tag{7.40}$$

From this equality and from evaluations (7.34) and (7.37) we come to

$$\frac{dw_+}{dx} = \frac{(k^2 - b) Ay_0 - dk^2 y_0 - dk^2 Ax - k^2 (k^2 - b) x + O(\delta^{3/2})}{y_0 - \gamma(x) + O(\delta^2)} \gamma(x). \tag{7.41}$$

Similarly, from equality (7.40) and evaluations (7.38) and (7.39), we obtain

$$\frac{dw_-}{dx} = \frac{-(k^2 - b) Ay_0 + dk^2 y_0 - dk^2 Ax - k^2 (k^2 - b) x + O(\delta^{3/2})}{-y_0 - \gamma(x) + O(\delta^{3/2})} \gamma(x). \tag{7.42}$$

Take the difference $\frac{dw_+}{dx} - \frac{dw_-}{dx}$. Directly from calculation it is easy to see the truth of the following relations:

$$\frac{dw_+}{dx} - \frac{dw_-}{dx}$$
$$= 2\,\frac{(k^2A - Ab - dk^2)\,\gamma(x) - k^2(k^2 - b + dA)\,x + O(\delta^{3/2})}{(y_0 - \gamma(x) + O(\delta^2))\,(y_0 + \gamma(x) + O(\delta^{3/2}))}\,\gamma(x)\,y_0. \tag{7.43}$$

This evaluation is true on path $\phi(p, t)$ for $0 \le x \le \epsilon$.

Now evaluate the increase of function w along path $\phi(p, t)$ for the passage of the final strip $0 \le x \le \delta$. By supposition we have

$$\left\{p\epsilon\ x = 0,\ |x_1| \le \delta^{3/2},\ \frac{1}{2}(A^2 + k^2) \le w \le 2(A^2 + k^2)\right\}.$$

However, from the definition of function w, we have

$$1 + 0(\delta) \leqslant y_0 \leqslant 2 + O(\delta). \tag{7.44}$$

Thus, from (7.43) and condition $E(h, H, \delta)$, for sufficiently small δ there is true the inequality

$$\frac{dw_+}{dx} - \frac{dw_+}{dx} > \frac{1}{2}\Big[(k^2A - Ab - dk^2)\,\gamma(x)$$
$$- k^2(dA + k^2 - b)\,x\big]\,\gamma(x) + O(\delta^{3/2})\,\gamma(x). \tag{7.45}$$

Accordingly, because by hypothesis, $Ehx \le \gamma(x) \le Hx$ for $0 < x \le \delta$, we obtain

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > \frac{1}{2}\,[(k^2A - Ab - dk^2)\,h$$
$$- k^2(dA + k^2 - b)]\,hx^2 + O(\delta^{5/2}) \tag{7.46}$$

for $x\epsilon[0, \delta]$. Integrating this inequality from $x = 0$ to $x = \delta$, we arrive at

$$\Delta_1 w > \frac{1}{6}\,[(k^2A - Ab - dk^2)\,h - k^2(dA + k^2 - b)]\,h\delta^3 + O(\delta^{7/2}), \tag{7.47}$$

where $\Delta_1 w$ is the increase of function w for the passage of path $\phi(p, t)$ on strip $0 \le x \le \delta$.

Next, evaluate the increase of function w along path $\phi(p, t)$ for the passage of the next strip $\delta \le x \le \epsilon$. From (7.43), (7.44) and condition $E(h, H, \delta)$ we obtain the inequality

$$\frac{dw_+}{dx} - \frac{dw_-}{dx} > -4\,k^2(dA + k^2 - b)\,x\,\gamma(x) + O(\delta^{5/2}). \tag{7.48}$$

This inequality is true on path $\phi(p, t)$ for $x\epsilon[\delta, \epsilon]$. Integrating inequality (7.48) from $x = \delta$ to $x = \epsilon$ we arrive at

$$\Delta_2 w > -\frac{4}{3}k^2(dA + k^2 - b)H(\varepsilon^3 - \delta^3) + O(\delta^{5/2})(\varepsilon - \delta). \tag{7.49}$$

Here $\Delta_2 w$ designates the increase of function w for a passage of path $\phi(p, t)$ on strip $\delta \le x \le \epsilon$. From inequality (7.49) and condition (7.8) we obtain

$$\Delta_2 w > O(\delta^4). \tag{7.50}$$

Finally, evaluate the increase of function w along path $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$. From formula (7.16), evaluation (7.17) and the definition of point p, all coordinates of path $\phi(p, t)$ on the interval of time $[0, t_1]$ are /169
bounded in absolute value by one and the same number which is dependent only on parameters a, b, c and d of system (7.10).

From the definition of the instants of time τ_1 and τ_2, $x \ge \epsilon$ on path $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$; but then from condition (7.7), $0 < \gamma(x) \le \delta^4$ on $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$. Accordingly, from (7.23) the relation

$$\dot{w} = 0(\delta^4) \tag{7.51}$$

is fulfilled on path $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$. Further, because of formula (7.16) and evaluation (7.17), $\tau_2 - \tau_1 < \frac{2\pi}{k}$. And then from (7.51) it follows that

$$\Delta_3 w = 0(\delta^4), \tag{7.52}$$

where $\Delta_3 w$ designates the increase of function w along path $\phi(p, t)$ on the interval of time $\tau_1 \le t \le \tau_2$. Combining relations (7.47), (7.50) and (7.52) we obtain the evaluation

$$\begin{aligned} \Delta w > \frac{1}{6}\,[(k^2 A - Ab - dk^2)\,h \\ - k^2(k^2 + dA - b)]\,h\delta^3 + O(\delta^{7/2}), \end{aligned} \tag{7.53}$$

where Δw is the increase of function w along path $\phi(p, t)$ for $t \epsilon [0, t_1]$. Thus, from condition (7.9) and inequalities (7.25) and (7.27), inequality (7.29) is fulfilled for sufficiently small δ.

Proceeding further with this same reasoning for $t > t_1$, we will prove inequality (7.28). And from this inequality, as mentioned above, the assertion of the lemma also follows.

Thus the lemma is proved.

The assertion of the following theorem results from what was just proved by the lemma.

Theorem 7.1

Suppose that the conditions of cases 8, 11, 12 or 13 are fulfilled; suppose moreover that function $\gamma(x) = f(x) - Ax$ satisfies condition $E(h, H, \delta)$ with a sufficiently small δ; then system (2.8) has solutions which do not go to the origin for $t \to +\infty$.

The assertion of this theorem follows from relation (7.26) and the meaning of transformation I.

Corollary

If the conditions of cases 8, 11, 12 or 13 are fulfilled, the null solution of system (2.8) is not globally stable for any f(x) satisfying the GHC (2.27).

Proof

We will show first that inequality

$$\frac{k^2(b-k^2-dA)}{Ab-Ak^2+dk^2}<B-A \tag{7.54}$$

takes place in the conditions of the cases considered. We have

$$b-k^2=b-c-dA=dB. \tag{7.55}$$

Thus we obtain

$$\frac{k^2(b-k^2-dA)}{Ab-Ak^2+dk^2}=\frac{k^2d(B-A)}{d(AB+k^2)}<B-A,$$

and then (7.54) also occurs.

Now we take an arbitrary function $\gamma(x) = f(x) - Ax$ satisfying condition $E(h, H, \delta)$ in which quantities h and H obey the inequality

$$B-A>H\geqslant h>\frac{k^2(b-k^2-dA)}{Ab-Ak^2+dk^2} \tag{7.56}$$

and δ is sufficiently small. Inequality (7.56) is contradictory. Furthermore, from the conditions of E, function $\gamma(x)$ thus chosen satisfies the GHC $0<\frac{\gamma(x)}{x}<B-A$ for $x \neq 0$.

Therefore, the assertion of the corollary also follows from theorem 7.1.

Remarks

In the conditions of E, h is an arbitrary number satisfying inequality (7.9); therefore, the region of variation of function $\gamma(x)$ given by theorem 4.16 cannot be broadened. That is, whatever the number $h_1 >$ $\frac{k^2(b-k^3-dA)}{Ab-Ak^2+dk^2}$, a function $\gamma(x)$ can be found which satisfies inequality

$$0 < \gamma(x)x < h_1x^2 \text{ for } x \neq 0 \tag{7.57}$$

such that system (2.8) will have a solution not going to origin for $t \to + \infty$.

Theorem 7.2

Suppose that the conditions of cases 14, 21 or 22 and inequality (7.24) are fulfilled. Suppose, moreover, that function $\gamma(x) = f(x) - Ax$ satisfies condition E with a sufficiently small δ. Then system (2.8) has a solution not going to the origin as $t \to + \infty$.

Corollary

Suppose that the conditions of either case 14, 21 or 22 and inequality (7.24) are fulfilled; then the null solution of system (2.8) is not globally stable for any functions f(x) satisfying the GHC.

Proof

If the conditions of case 21 are fulfilled, then the assertion of the corollary is obvious since any function $\gamma(x)$ satisfying condition E(h, H, δ) also satisfies the GHC $\frac{\gamma(x)}{x} > 0$ for $x \neq 0$.

We will show that the inequality

$$\frac{k^2(b - k^2 - dA)}{Ab - Ak^2 + dk^2} < -\frac{a}{b} - A \tag{7.58}$$

takes place for the fulfillment of the conditions of case 14 or 22. From the definition of quantities A and k^2, we have

$$-\frac{a}{b} - A = -\frac{Ak^2}{b}, \tag{7.59}$$

therefore inequality (7.58) is rewritten as

$$\frac{k^2(b - k^2 - dA)}{Ab - Ak^2 + dk^2} < -\frac{Ak^2}{b}. \tag{7.60}$$

Since $b < 0$ in the conditions of the cases considered, from (7.60), because of (7.25), we obtain

$$k^2b^2 - k^4b - dAk^2b < - A^2bk^2 + A^2k^4 - Adk^4$$

or

$$k^2(A^2 - dA + b)(b - k^2) < 0. \tag{7.61}$$

The last inequality results from $b < 0$ and inequality (7.24). Thus, inequality (7.58) is proven. From this inequality, h and H can be chosen such that function $\gamma(x)$ satisfying the conditions $E(h, H, \delta)$, with a δ as small as is desirable, will at the same also satisfy the generalized Hurwitz condition $0 < \frac{\gamma(x)}{x} < -\frac{a}{b} - A$ for $x \neq 0$. Therefore, the assertion of the corollary follows from theorem 7.2.

Remark

From theorem 7.2 we see that the region of variation of function $\gamma(x)$ given by theorem 4.17 cannot be widened; i.e., whatever the value $h_1 > \frac{k^2(b - k^2 - dA)}{Ab - Ak^2 - dk^2}$, a function $\gamma(x)$ can be found satisfying inequality (7.57) such that system (2.8) will have solutions not going to origin for $t \to +\infty$.

Theorem 7.3

Suppose that the conditions of case 16 and inequality (7.25) are fulfilled. Suppose, moreover, that function $\gamma(x) = f(x) - Ax$ satisfies condition $E(h, H, \delta)$ for sufficiently small δ. Then system (2.8) has a solution not going to the origin for $t \to +\infty$.

Corollary

If the conditions of case 16 and inequality (7.25) are fulfilled, then the null solution of system (2.8) is not globally stable for any $f(x)$ satisfying the generalized Hurwitz condition (2.35).

Remark

In consequence of theorem 7.3, the range of the function given by theorem (4.18) cannot be broadened; i.e., whatever the value of the number

$$h_1 > \frac{k^2(b - k^2 - dA)}{Ab - Ak^2 + dk^2},$$

a function $\gamma(x)$ can be found satisfying inequality (7.57) such that system (2.8) has solutions not going to the origin for $t \to +\infty$.

Section 21

In this section we will formulate some conditions for the existence of periodic solutions to system (2.8) for $d \neq 0$.

Theorem 7.4

Let the conditions of either case 8, 11, 12 or 13 be fulfilled. Further, let function $\gamma(x) = f(x) - Ax$ satisfy condition $E(h, H, \delta)$ with sufficiently small δ. Suppose, moreover, that a positive λ exists such that

$$\gamma(x) \cdot x \geq \lambda \text{ for } |x| \geq \epsilon, \tag{7.62}$$

where ϵ is the number figuring in condition E. Then system (2.8) has periodic solutions different from the equilibrium position.

Proof

As earlier, use p to designate domain $\{x = 0,\ |x_1| < \delta^{3/2},\ w > \frac{1}{2}(A^2 + k^2),\ y > 0\}$, where w is the function defined by equality (7.22). Further, introduce function v for consideration in the formula

$$\begin{aligned} v &= \frac{1}{2} x_1^2 + \frac{1}{2} \nu y_1^2 + \frac{1}{2} \nu z_1^2 \\ &+ \nu (bA - Ak^2 + dk^2) \int_0^x \gamma(x)\, dx, \end{aligned} \tag{7.63}$$

where

$$\nu = \frac{A^2 - dA + b}{k^2 - b}. \tag{7.64}$$

It is easy to see that $\nu > 0$ in the considered cases. Because of the differential equations of system (7.10), the time derivative of function v is equal to

$$\begin{aligned} \dot{v} = &- Ax_1^2 - \nu [k^2 (k^2 - b + dA)\, x \\ &- (Ak^2 - Ab - dk^2)\, \gamma(x)]\, \gamma(x). \end{aligned} \tag{7.65}$$

Take an arbitrary point $p \epsilon \bar{P}$ and let $0, t_1, t_2$ be a sequence of instants of the intersection of path $\phi(p, t)$ with plane $x = 0$. As proved above, on the interval of time $0 < t < t_1$, there exist only two instants of the intersection of path $\phi(p, t)$ with plane $x = \epsilon$. Let τ_1 and τ_2 be these instants; we will then say that $\tau_1 < \tau_2$.

Because of formula (7.16) and evaluation (7.17) the inequality

$$(Ak^2 - Ab - dk^2)H^2\varepsilon^2(\tau_1 + t_1 - \tau_2) < \frac{\lambda k^2}{2}(k^2 - b - dA)(\tau_2 - \tau_1) \tag{7.66}$$

will be filled on path $\phi(p, t)$ if a sufficiently large $v_0 > 2(A^2 + k^2)$ is taken and if only point p lies on that part of curve $\{x = 0, v = v_0\}$ which is disposed of in the closed domain $\overline{P}$. Designate this arc by 1. Let p be an arbitrary point of arc 1; we will calculate the increase Δv, of function v along path $\phi(p, t)$ for a variation of t from 0 to t_1. From formula (7.65) and condition E, the inequality

$$\dot{v} < \nu(Ak^2 - Ab - dk^2)H^2\epsilon^2 \tag{7.67}$$

is fulfilled on $\phi(p, t)$ for $t \epsilon [\tau_2, t_1]$ and $t \epsilon [0, \tau_1]$. Designate by $\Delta_1 v$ the greater increase in function v along path $\phi(p, t)$ on the intervals of time $[0, \tau_1]$ and $[\tau_2, t_1]$; then from (7.67) we obtain

$$\Delta_1 v < \nu(Ak^2 - Ab - dk^2)H^2\varepsilon^2(\tau_1 + t_1 - \tau_2). \tag{7.68}$$

From equality (7.65) and condition $E(h, H, \delta)$, for a sufficiently small δ on path $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$, there is fulfilled the inequality

$$\dot{v} < -\frac{\nu k^2}{2}(k^2 - b + dA)x\gamma(x). \tag{7.69}$$

Consequently, from condition (7.62) of the theorem to be proved, on $\phi(p, t)$ for $t \epsilon [\tau_1, \tau_2]$ there takes place the inequality

$$\dot{v} < -\frac{\nu k^2 \lambda}{2}(k^2 - b + dA). \tag{7.70}$$

Let $\Delta_2 v$ be the increase of v on the interval of time $[\tau_1, \tau_2]$. Integrating inequality (7.70) along $\phi(p, t)$ from $t = \tau_1$ for $t = \tau_2$, we have

$$\Delta_2 v < -\frac{\nu k^2 \lambda}{2}(k^2 - b + dA)(\tau_2 - \tau_1). \tag{7.71}$$

By adding inequalities (7.68) and (7.71) and because of (7.66), we obtain

$$\Delta v < 0. \tag{7.72}$$

The last inequality means that

$$v(p) > v(\phi(p, t_1)). \tag{7.73}$$

Using the same reasoning for path $\phi(p, t)$ for a variation of t from t_1 to t_2, we obtain the inequality

$$v(p) > v(\phi(p, t_2)). \tag{7.74}$$

Designate by Q the domain $\{x = 0,\ |x_1| < \delta^{3/2},\ v < v_0,\ w > \frac{1}{2}(A^2 + k^2),\ y > 0\}$, and by $\overline{Q}$ its closure. As earlier, we will designate by I the transformation of the closed domain $\overline{P}$ into itself for motions of the path of system (2.8). From lemma 7.3 and relation (7.74), we have

$$I(\overline{Q}) \subset Q. \tag{7.75}$$

Therefore, because of the well-known theorem of Brouwer, in domain Q there is a point q fixed relative to transformation I. However, from the meaning of transformation I, path $\phi(q, t)$ is a path of a periodic motion of system (2.8).

Thus the theorem is proved.

Theorem 7.5

Suppose that the conditions of either case 14, 21 or 22 and inequality (7.24) are fulfilled. Further, let function $\gamma(x) = f(x) - Ax$ satisfy condition $E(h, H, \delta)$ with sufficiently small δ. Suppose moreover that there exists a $\lambda > 0$ such that condition (7.62) is fulfilled. Then system (2.1) has a periodic solution different from the equilibrium position.

Theorem 7.6

Let the conditions of case 16 and inequality (7.25) be fulfilled. Let function $\gamma(x) = f(x) - Ax$ satisfy condition $E(h, H, \delta)$ with sufficiently small δ. Suppose, moreover, that a $\lambda > 0$ exists such that condition (7.62) is fulfilled. Then system (2.8) has a periodic solution different from the equilibrium position.

The last two theorems are proved in the same manner as was theorem 7.4.

Section 22

In this section we will consider case 17 and establish the necessity of the conditions of theorem 4.19. For the proof of the necessity of conditions (4.188) and (4.189), suppose that at least one of these conditions is not fulfilled, and we will show that then there is a path going to infinity for increasing time for system (2.8). For definiteness let

$$\overline{\lim_{x \to +\infty}} \left(\gamma(x) + \int_0^x \gamma(x)\, dx \right) < +\infty, \tag{7.76}$$

where, as earlier, $\gamma(x) = f(x) - Ax$.

Due to relation (7.76) there exists a number $M > 0$ such that

$$\gamma(x) < M \text{ for } x \geq 0. \tag{7.77}$$

Assume that

$$\int_0^{+\infty} \gamma(x)\, dx = J < +\infty. \tag{7.78}$$

We will consider two cases.

I. $G = A^2 - dA + b \geq 0$. In this case consider point p with coordinates $x = 0$, $y = y_0$, $z = z_0$.

Let the coordinates of point p satisfy the inequalities

$$Ay_0 > z_0 > 2AM + \frac{bJ}{M}. \tag{7.79}$$

As earlier, let $x_1 = A^2x - Ay + z$. Then

$$\frac{dx_1}{dt} = -Ax_1 - G\gamma(x).$$

From the last equality we see that on path $\phi(p, t)$ there is fulfilled the equality

$$x_1 = e^{-At}\left[x_{10} - G\int_0^t \gamma \cdot e^{At} dt\right], \tag{7.80}$$

where $x_{10} = z_0 - Ay_0$.

We will show that on path $\phi(p, t)$ for $t \geq 0$ there is fulfilled the inequality

$$z > 2AM. \tag{7.81}$$

Indeed, for $t = 0$ this inequality is fulfilled as follows from (7.79).

Suppose that it is violated for $t = t_1 > 0$, and let t_1 be the first instant after $t = 0$ for which inequality (7.81) is violated; i.e., suppose that the following relations are fulfilled:

$$z(\phi(p, t_1)) = 2AM \tag{7.82}$$

and

$$z(\phi(p, t)) > 2AM \text{ for } t \in [0, t_1).$$

We will show that then

$$y - Ax > \frac{z}{A}. \tag{7.83}$$

on path $\phi(p, t)$ for $t \epsilon [0, t_1]$. Indeed, for $t = 0$ this inequality is fulfilled. Now assume that there exists a $t^* \epsilon (0, t_1]$ such that on path $\phi(p, t)$ it occurs that

$$y(t^*) - Ax(t^*) = \frac{z(t^*)}{A}, \tag{7.84}$$

and inequality (7.83) is fulfilled on $\phi(p, t)$ for $t \epsilon [0, t^*)$. But for $t \epsilon [0, t_1]$, $z \geq 2AM$ by supposition; consequently,

$$y - Ax \geq 2M > \gamma(x)$$

on path $\phi(p, t)$ for $t \epsilon [0, t^*]$. Then for such t, x monotonically increases along path $\phi(p, t)$; and therefore, $x > 0$ for $t \epsilon (0, t^*]$ on $\phi(p, t)$. However, by hypothesis, $G \geq 0$; from this and from (7.80), $x_1 < 0$ occurs for $t \epsilon [0, t^*]$ on trajectory $\phi(p, t)$. And this also so that inequality (7.83) is fulfilled for $t \epsilon [0, t^*]$, which contradicts equality (7.84). The contradiction obtained thus proves that inequality (7.83) is fulfilled on path $\phi(p, t)$ for all $t \epsilon [0, t_1]$.

However, from equality $\frac{dz}{dx} = \frac{-b\gamma(x)}{y - Ax - \gamma(x)}$ we obtain

$$\frac{dz}{dx} > \frac{-b\gamma(x)}{y - Ax - M} \tag{7.85}$$

on $\phi(p, t)$ for $t \epsilon [0, t_1]$. And thus, from (7.82) and (7.83) we have

$$\frac{dz}{dx} > -\frac{b\gamma(x)}{M} \tag{7.86}$$

on path $\phi(p, t)$ for $t \epsilon [0, t_1]$. Integrating this inequality along path $\phi(p, t)$ on the interval of time $0 \leq t \leq t_1$, we obtain

$$z(t_1) - z_0 > -\frac{bJ}{M}.$$

Accordingly, from (7.79) we establish that

$$z(t_1) > 2AM. \tag{7.87}$$

This inequality contradicts relation (7.82). The contradiction obtained thus proves that inequality (7.81) is fulfilled on path $\phi(p, t)$ for all $t \geq 0$.

Since $A > 0$ and $M > 0$, from inequality (7.81) path $\phi(p, t)$ does not go to the origin. Consequently, global stability is absent in the case considered.

II. $G < 0$. In this case, consider point p with coordinates $x = 0$, $y = y_0$, $z = z_0$, and let

$$z_0 = Ay_0 > 2AM + (b - G)\frac{J}{M} \tag{7.88}$$

We will show that the inequality

$$y - Ax > 2M \tag{7.89}$$

is fulfilled on path $\phi(p, t)$ for $t \geq 0$. Indeed, this inequality is fulfilled for $t = 0$, as follows from (7.88).

Suppose that it is violated for $t = t_1 > 0$, and let t_1 be the first instant after $t = 0$ for which inequality (7.89) is not fulfilled; i.e., suppose that on path $\phi(p, t)$ there results

$$y(t_1) - Ax(t_1) = 2M, \tag{7.90}$$

and inequality (7.89) is fulfilled on $\phi(p, t)$ for $t \epsilon [0, t_1)$.

From equality $\frac{dz}{dx} = \frac{-b\gamma(x)}{y - Ax - \gamma(x)}$ we obtain

$$\frac{dz}{dx} > \frac{-b\gamma(x)}{y - Ax - M}$$

and thus, because of (7.90), we arrive at

$$\frac{dz}{dx} > -\frac{b\gamma(x)}{M} \tag{7.91}$$

on path $\phi(p, t)$ for $t \epsilon [0, t_1]$. Integrating this inequality along path $\phi(p, t)$ on the interval of time $0 \leq t \leq t_1$, we have

$$z(t_1) - z_0 > -\frac{bJ}{M}. \tag{7.92}$$

And thus, from (7.88) we obtain

$$z(t_1) > 2AM - \frac{GJ}{M}. \tag{7.93}$$

Now return again to equality $\frac{dx_1}{dt} = -Ax_1 - G\gamma(x)$. From this equality and because of (7.88), we obtain for path $\phi(p, t)$

$$x = -e^{-At}G\int_0^t \gamma \cdot e^{At}dt. \tag{7.94}$$

Consequently, from (7.89), as in the preceding case, we establish that inequality $x_1 \geq 0$ is fulfilled on path $\phi(p, t)$ for $t\epsilon[0, t_1]$.

Divide equality $\frac{dx_1}{dt} = -Ax_1 - G\gamma(x)$ by the first equation of system (2.8) to obtain

$$\frac{dx_1}{dx} = \frac{-Ax_1 - G\gamma(x)}{y - Ax - \gamma(x)}.$$

Accordingly, because $x_1 \geq 0$ on path $\phi(p, t)$ for $t\epsilon[0, t_1]$, we obtain

$$\frac{dx_1}{dx} \leqslant -\frac{G\gamma(x)}{y - Ax - \gamma(x)}$$

on $\phi(p, t)$ for $t\epsilon[0, t_1]$.

Due to (7.89), the last inequality gives

$$\frac{dx_1}{dx} \leqslant -\frac{G\gamma(x)}{M} \tag{7.95}$$

on path $\phi(p, t)$ for $t\epsilon[0, t_1]$.

By integrating the last inequality, we obtain

$$x_1(t_1) - x_1(0) < -\frac{GJ}{M}.$$

However, $x_1(0) = 0$ because of (7.88). Therefore, for path $\phi(p, t)$, from the last inequality we obtain

$$A(Ax(t_1) - y(t_1)) + z(t_1) < -\frac{GJ}{M}. \tag{7.96}$$

Thus, from (7.93) we obtain

$$A(Ax(t_1) - y(t_1)) + 2AM - \frac{GJ}{M} < -\frac{GJ}{M}$$

or $y(t_1) - Ax(t_1) > 2M$ on path $\phi(p, t)$. The last inequality contradicts relation (7.90). The contradiction obtained thus proves that relation (7.89) is fulfilled on path $\phi(p, t)$ for $t \geq 0$. Accordingly, path $\phi(p, t)$ does not go to the origin. Consequently, global stability is not present in this case.

Thus the following theorem is established.

Theorem 7.7

Suppose that the conditions of case 17 are fulfilled; then, for the trivial solution of system (2.8) to be globally stable in the whole, it is necessary and sufficient that the following conditions be fulfilled:

$$\overline{\lim_{x\to+\infty}}\left(\gamma(x)+\int_0^x \gamma(x)\,dx\right)=+\infty$$

and

$$\overline{\lim_{x\to-\infty}}\left(-\gamma(x)+\int_0^x \gamma(x)\,dx\right)=+\infty.$$

Corollary

If the conditions of case 17 are fulfilled, then the null solution of system (2.8) is not globally stable for any f(x) satisfying the generalized Hurwitz conditions (2.36).

CONCLUSIONS

The problem which is solved in the present work consists of the following. For what values of the parameters a_{ij} is the null solution of the system of three differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x) + a_{12}y + a_{13}z, \\ \frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z, \\ \frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z. \end{aligned}\right\} \tag{2.1}$$

globally stable for any nonlinearity f(x) satisfying the generalized Hurwitz conditions? In other words, it is required to ascertain for what values of a_{ij} is the question of Ayzerman's problem answered in the affirmative. This problem is studied in this work in a comprehensive way. The following results are obtained.

If $a_{22} + a_{33} = 0$, system (2.1), by a change of variables, is put into the form

$$\frac{dx}{dt} = y - f(x), \quad \frac{dy}{dt} = z - x, \quad \frac{dz}{dt} = -ax - bf(x). \tag{2.15}$$

For the null solution of system (2.15) to be globally stable for any nonlinearity f(x) satisfying the generalized Hurwitz conditions, it is necessary and sufficient that one of three conditions be fulfilled:

$$\begin{aligned} &\text{I. } a < 0, \quad b > 0, \\ &\text{II. } a = 0, \quad 0 < b < 1, \\ &\text{III. } a > 0, \quad b < 0, \quad \frac{a^2}{(1-b)^2} + b \leqslant 0. \end{aligned}$$

But if $a_{22} + a_{33} \neq 0$, then system (2.1) is put into the form

$$\frac{dx}{dt} = y - f(x), \quad \frac{dy}{dt} = z - cx - df(x), \quad \frac{dz}{dt} = -ax - bf(x). \tag{2.8}$$

For the null solution of system (2.8) to be globally stable for any nonlinearity f(x) satisfying the generalized Hurwitz conditions, it is necessary and sufficient that one of the following conditions be fulfilled:

I. The conditions of cases 9, 10, 15, 18, 19 or 20.

II. The conditions of cases 14, 21 or 22 and the inequality

$$A^2 - Ad + b \leq 0.$$

III. The conditions of case 16 with inequality

$$Ab - Ak^2 + dk^2 \geq 0.$$

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